

Minimum Time Trajectory Optimization for Flying a Quadrotor in an 8-shaped Path

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Abstract

In this paper, minimum time control problem is formulated for a quadrotor to go around two focal points in an 8-shaped path. Then the optimal control problem is reduced to a simpler form using point mass model and numerical results are presented. The requirement of path to be an 8-shaped, is translated to state constraints using a special geometry definition. Total flight time of 3.5 seconds is obtained for a single 8-shaped path. The resulted trajectory can be used as the reference input for practical control applications.

1 Introduction

Quadrotors have become more popular in the last few decades for a variety of applications in the military and civilian fields. Especially battery energized quadrotors have the benefits of small size, portable and low noise flying capability. In such MAVs optimal control is an important issue based on the mission requirements. For example energy, average velocity or time optimization may be a key feature in control strategy.

Some recent works are reviewed here. An algorithm for finding minimum time paths through any number of waypoints in R^3 subject to hardware actuator saturation is presented in [3] for a variable pitch quadrotor. In [6] the optimal trajectory is generated such that the swing of a suspended payload is minimized for a maneuvering quadrotor. Trajectory generation with minimum snap (the second derivative of acceleration) for tightly constrained indoor environments is solved in [5]. Optimal path planning derived in [2] bases on running cost optimization by meaning of minimizing average velocity during the flight time of the quadrotor. In [4] time-optimal motion planning problem (TOMP) is solved for a hovering quadrotor which initial and final velocities must be zero.

In this paper the TOMP problem is solved for a maneuvering quadrotor for cruising around two focal points on a self crossed (8-shaped) path. After dynamic modeling in the following sections, the optimal control problem is formulated, and then is reduced to a simpler form and finally numerical results are presented.

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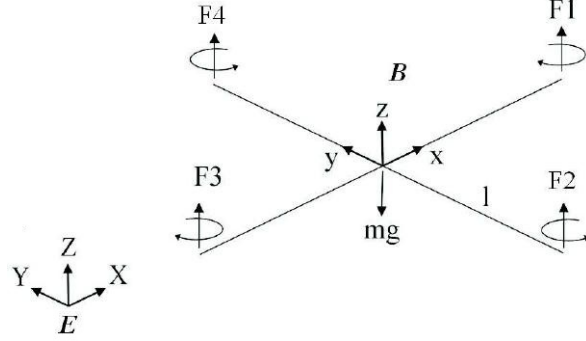


Figure 1: Quadrotor configuration, B: body fixed frame, E: inertial frame.

2 Dynamic Model of a Quadrotor

A quadrotor has four propellers in cross configuration as shown in figure 1. Two pairs of propellers (1,3) and (2,4) turn in opposite directions which neutralizes the counter-torque of propellers. The lift force can be changed by varying the rotor speed, so increasing or decreasing the speed of four propellers together leads to vertical motion. Changing 2 and 4 propeller's speed inversely produces roll rotation. Pitch rotation results when 1 and 3 propeller's speed inversely change. Roll and pitch rotations are coupled with lateral motion in y and x directions of body frame. Yaw rotation results from the difference in the counter-torque between each pair of propellers.

Consider earth fixed frame E and body fixed frame B, as seen in figure 1. This paper assumes the following points for obtaining motion equations:

- The structure and propellers are supposed to be rigid.
- The structure is symmetrical and the body fixed frame origin located on CG.
- Propeller's thrust and drag are proportional to the square of its rotational speed.
- Body drag due to motion and rotation is neglected due to low movement speed.
- Roll and pitch angles assumed to be small, thus the derivatives of Euler angles $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ are approximated by the body rotational speeds (p, q, r) .
- The motor dynamics is neglected.

External force and torque vectors applied on the body are obtained as:

$$F_B = \begin{bmatrix} 0 \\ 0 \\ b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \end{bmatrix}, M_B = \begin{bmatrix} lb(\Omega_4^2 - \Omega_2^2) \\ lb(\Omega_3^2 - \Omega_1^2) \\ d(\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2) \end{bmatrix} \quad (1)$$

The parameter description is summarized in table 1. Physical values are acquired from [1] for a typical quadrotor. Newton's second law for translational directions can be written as follow:

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R_{EB} \begin{bmatrix} 0 \\ 0 \\ b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \end{bmatrix} \quad (2)$$

where R is the rotational transformation matrix between earth frame and body frame. This matrix can be obtained based on Euler angles:

$$R_{EB} = R_\psi R_\theta R_\phi = \begin{bmatrix} \cos \psi \cos \theta & -\sin \psi \cos \phi + \cos \psi \sin \theta \sin \phi & \sin \psi \sin \phi + \cos \psi \sin \theta \cos \phi \\ \sin \psi \cos \theta & -\cos \psi \cos \phi + \sin \psi \sin \theta \sin \phi & -\cos \psi \sin \phi + \sin \psi \sin \theta \cos \phi \\ -\sin \theta & \cos \theta \sin \phi & \cos \theta \cos \phi \end{bmatrix}$$

When body rotates along the x and y directions of body frame, the gyroscopic moments due to propeller's rotation acts on the vehicle. These moments can be derived as:

$$M_G = J_r \vec{\omega} \times \vec{\Omega} = J_r \vec{\omega} \times \hat{b}_3 (\sum \Omega_i) = J_r \sum \Omega_i \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = J_r (\Omega_1 + \Omega_3 - \Omega_2 - \Omega_4) \begin{bmatrix} \dot{\theta} \\ -\dot{\phi} \\ 0 \end{bmatrix} \quad (3)$$

where $\vec{\omega}$ is the vector of body rotational speeds which approximated by the derivatives of Euler angles and $\hat{b}_3$ is the unit vector of body frame in z direction. Now the rotational form of Newton's second law can be applied as follow:

$$\begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \times \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = M_B + M_G \quad (4)$$

where M_B and M_G were expressed in equations 1 and 3. Finally the quadrotor model will be derived by solving equations 2 and 4 for second derivatives of position and angular states:

$$\begin{cases} \ddot{x} = (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) \frac{1}{m} U_1 \\ \ddot{y} = (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) \frac{1}{m} U_1 \\ \ddot{z} = -g + (\cos \phi \cos \theta) \frac{1}{m} U_1 \\ \ddot{\phi} = \dot{\theta} \dot{\psi} \frac{I_y - I_z}{I_x} + \frac{J_r}{I_x} \dot{\theta} \Omega + \frac{1}{I_x} U_2 \\ \ddot{\theta} = \dot{\phi} \dot{\psi} \frac{I_z - I_x}{I_y} - \frac{J_r}{I_y} \dot{\phi} \Omega + \frac{1}{I_y} U_3 \\ \ddot{\psi} = \dot{\phi} \dot{\theta} \frac{I_x - I_y}{I_z} + \frac{1}{I_z} U_4 \end{cases} \quad (5)$$

The system inputs can be obtained as:

$$\begin{cases} U_1 = b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \\ U_2 = b(\Omega_4^2 - \Omega_2^2) \\ U_3 = b(\Omega_3^2 - \Omega_1^2) \\ U_4 = d(\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2) \\ \Omega = \Omega_1 + \Omega_3 - \Omega_2 - \Omega_4 \end{cases} \quad (6)$$

3 Problem Formulation

Optimal control problem is formulated in this section. Such a problem consists of a mathematical model, a cost function and some constraints on control or state variables. The cost function is the whole flight time which should be minimized. Mathematical model was extracted in last section and will be modified in this section. Problem constraints are explained in the following section.

Symbol	Definition	Value	Unit
x, y, z	<i>Absolute position of CoG</i>	—	m
θ, ϕ, ψ	<i>Euler angles</i>	—	rad
Ω_i	<i>Angular speed of i-th propeller</i>	—	rad/s
m	<i>Quadrotor mass</i>	0.650	kg
g	<i>Gravity acceleration</i>	9.81	m/s^2
I_x, I_y	<i>Inertia moment on x or y axis</i>	$7.5E - 3$	$kg.m^2$
I_z	<i>Inertia moment on z axis</i>	$1.3E - 2$	$kg.m^2$
J_r	<i>Rotor inertia</i>	$6E - 5$	$kg.m^2$
l	<i>Arm length</i>	0.23	m
b	<i>Propeller Thrust factor</i>	$3.13E - 5$	$N.s^2$
d	<i>Propeller drag factor</i>	$7.5E - 7$	$N.m.s^2$

Table 1: Symbols definition and physical values

3.1 Constraints Definition

Problem constraints including physical limitations of quadrotor and the requirements of the desired path are explored here.

3.1.1 Control Constraints

The major physical limitation of a quadrotor is the propeller's rotational speed. This speed in hovering condition is obtained by solving equations 5 and 6 for equilibrium point:

$$\Omega_H = \sqrt{\frac{mg}{4b}} = 225.7 \quad rad/s \quad (7)$$

Assuming that maximum thrust is two times the MAV weight, saturation speed of the propellers should be approximately 41% higher than the hovering speed. Thus the control constraint is defined as:

$$0 \leq \Omega_i \leq 320 \quad rad/s \quad (8)$$

3.1.2 State Constraints

Constraints on state variables should be defined according to path requirements. Figure 2 presents a desired path similar to figure-8 in a horizontal plane. The requirement of path to be within the selected area can be defined by non-equality constraints, but the self crossing is a qualitative requirement which is necessary to translate to a mathematical form. Because the path is symmetric, one method is breaking the path into two closed loop half-paths around each focal point. Then the requirement of half-path to be closed loop can be addressed by defining suitable boundary conditions in a polar coordination system. The polar frame origin is located on one of the focal points. Suppose that the initial and final points of each half-path are located on the full-path center of symmetry (O). Thus the boundary conditions are defined as $\alpha(0) = -\pi$, $\alpha(t_f) = +\pi$ and $r(0) = r(t_f) = d/2$ which lead to a path around the origin by two reasons:

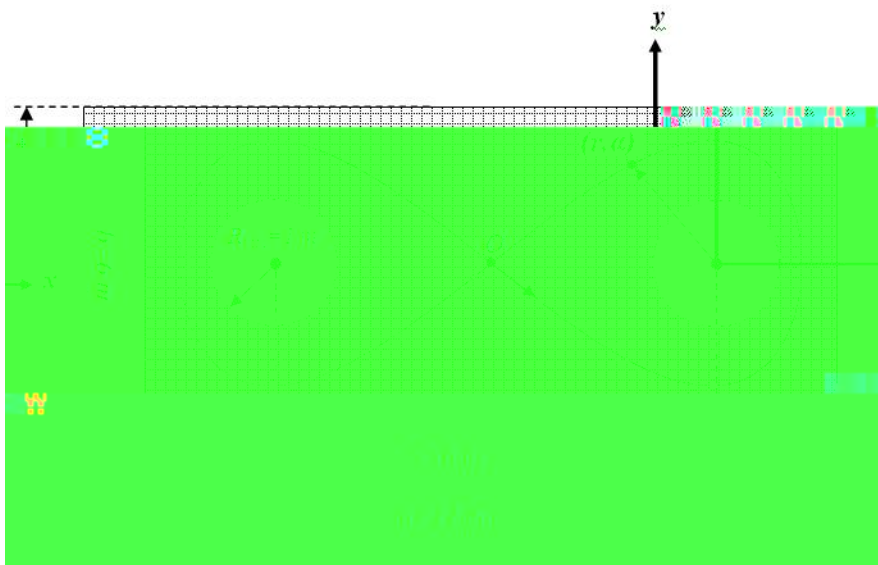


Figure 2: Conditions and constraints of desired path.

Constraint qualitative definition	Mathematical representation
<i>Flying within the rectangular area</i>	$-\frac{d}{2} \leq r \cos \alpha \leq \frac{a-d}{2}$ $-\frac{b}{2} \leq r \sin \alpha \leq \frac{b}{2}$
<i>Avoiding entering the prohibited circle</i>	$r \geq R_{min}$
<i>Fly in horizontal plane</i>	$\ddot{z} = 0$
<i>Closed loop around a focal point</i>	$r(0) = \frac{d}{2}, \quad \alpha(0) = -\pi$ $r(t_f) = \frac{d}{2}, \quad \alpha(t_f) = +\pi$

Table 2: Qualitative and mathematical definition of constraints and boundary conditions

- First the angle should change smoothly from $-\pi$ to $+\pi$, thus it is forced to cross a zero angle position.
- Second the origin is located in the prohibited area, thus the MAV in the zero angle position must satisfy a minimum radius of 1 meter.

Therefore these boundary conditions will lead the MAV to go around the origin and make a closed loop half-path. As a conclusion for this section, the qualitative and mathematical definition of the path requirements are summarized in table 2.

3.2 Equations in Cylindrical Coordination System

The necessity of dynamics modeling in cylindrical coordination system (r, α, z) displayed in last section. By definition:

$$\begin{cases} x = r \cos \alpha \\ y = r \sin \alpha \\ z = z \end{cases} \quad (9)$$

the second derivatives of coordination parameters derived as:

$$\begin{cases} \ddot{r} = r\dot{\alpha}^2 + \ddot{x} \cos \alpha + \ddot{y} \sin \alpha \\ \ddot{\alpha} = (-2\dot{r}\dot{\alpha} - \ddot{x} \sin \alpha + \ddot{y} \cos \alpha) \frac{1}{r} \\ \ddot{z} = \ddot{z} \end{cases} \quad (10)$$

3.3 Problem Setup

Optimal control problem is summarized here. The goal is to find an optimal trajectory for a quadrotor with the mathematical model defined by equations 10, 5 and 6, which minimizes the whole flight time ($J = \int_0^{t_f} dt$, $t_f : free$) subject to constraints described by equation 8 and table 2 assuming boundary values represented in table 2.

4 Solution

The presented model in last section is very difficult to solve due to its high complexity and nonlinearity. On the other hand, a simpler model is sufficient for path planning applications. In the following a simpler problem is derived after some considerations and then the numerical results are presented.

4.1 Problem Simplification

There are two strategies to flight a quadrotor in an 8-shaped path:

- Variable heading angle according to tangent direction of the path.
- Fixed heading angle.

The advantage of first strategy highlights when the MAV uses a front camera to monitor videos for a pilot. In this case the time response of the heading angle limits the circulation speed. A physical sense leads that the second strategy has a higher performance for autonomous flying because it utilizes lower mobility functions of the system. Although this sentence should be verified in future works, but used in this paper as a result. Thus the heading angle is assumed to be zero ($\psi = 0$).

Model reduction is used as the second simplification in this paper. Many quadrotors have internal control loops using PID controllers for flight stability and then accept some guidance commands including throttle, roll, pitch and yaw angles from ground station via wireless communication modules. This is possible to derive a 3DOF point mass model for such a quadrotor. In this case the system inputs are T, ϕ, θ, ψ and the rotational dynamics are neglected. According to equations 5 and 10,

state space model of the system results as:

$$\dot{q} = \begin{bmatrix} \dot{r} \\ \dot{\alpha} \\ \dot{z} \\ \ddot{r} \\ \ddot{\alpha} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ \dot{\alpha} \\ \dot{z} \\ r\dot{\alpha}^2 + \frac{T}{m}(\cos \phi \sin \theta \cos \alpha - \sin \phi \sin \alpha) \\ -\frac{2\dot{r}\dot{\alpha}}{r} - \frac{T}{mr}(\cos \phi \sin \theta \sin \alpha + \sin \phi \cos \alpha) \\ -g + \frac{T}{m} \cos \phi \cos \theta \end{bmatrix} \quad (11)$$

Constraints on control inputs defined as bellow:

$$\begin{cases} 0 \leq T \leq 2mg \\ -60 \leq \phi, \theta \leq 60 \quad Deg \end{cases} \quad (12)$$

4.2 Solution Method

Optimal control problem is solved by using the GPOPS (General-Purpose Pseudospectral Optimal Control Software) which is an open source MATLAB code. GPOPS employs an hp-adaptive Radau pseudospectral Gaussian quadrature method where the collocation is performed at the Legendre-Gauss-Radau quadrature points [7].

An important matter affecting the setting of boundary conditions is the difference between first and other repetitive loops. In this paper the primary path is ignored including take off and joining to the optimal path. After taking off and pure climbing, the MAV joins the optimal path at point "O" with zero speed for starting the first loop. But in the next loops the initial speed equals the final speed of the last loop. This paper's goal is obtaining the steady state trajectory for a long time mission which will be used as the reference input for closed loop controller. For achieving this requirement, the initial and final speed of the half-path assumed to be equal and free, because the trajectory is symmetric.

4.3 Numerical Results

The optimization problem is solved by the described method in last section and the numerical results are presented here. The Optimal path is shown in figure 3 and the control inputs are displayed in figure 4. According to this figure, a single 8-shaped path can be done in approximately 3.5 seconds with this autonomous method. Another result is that the optimal path passes as possible closest to the prohibited circles.

5 Conclusions and future works

Optimal control problem was formulated based on a 6DOF model of the quadrotor and then reduced to a simpler form by considering the fixed heading angle strategy and 3DOF point mass model for a typical quadrotor that accepts some control commands including throttle, roll and pitch angles. The requirement of path to be an 8-shaped around two poles was translated to a set of suitable boundary conditions in cylindrical coordination system. Final numerical results including optimal path curve and control inputs as functions of time were presented. The minimum flight time for a single 8-shaped path was obtained as 3.5 seconds and the optimal path is as possible close to the prohibited area. Some suggestions are itemized here for future works:

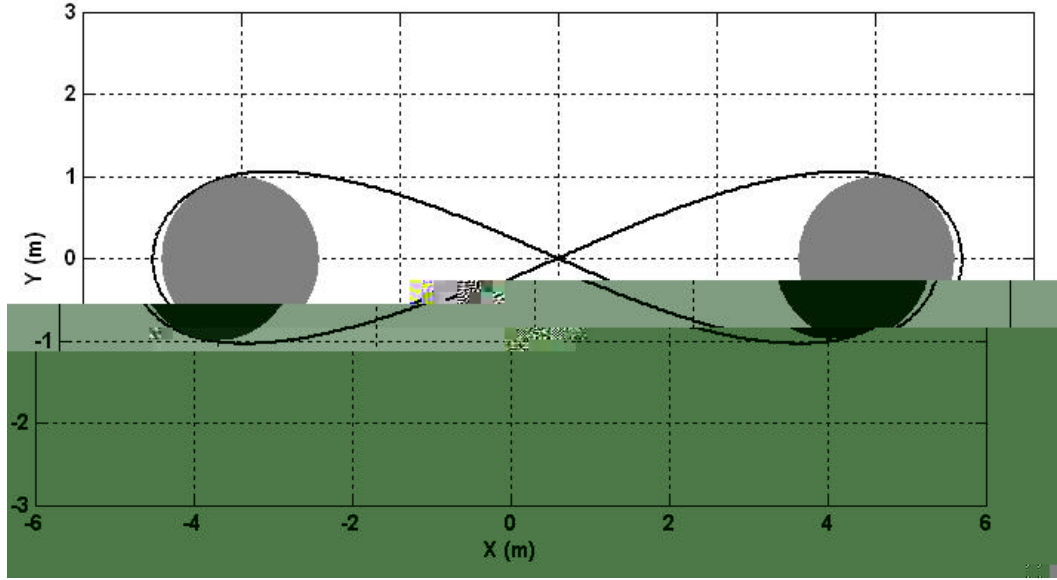


Figure 3: Optimal path in x-y plane.

- Verifying the better performance of fixed rather than variable heading angle strategy.
- Solving the problem for full 6DOF model and comparing the results with this paper.
- Implementing the controller for a quadrotor based on the resulted optimal trajectory and comparing the average cycling time with other control strategies.

References

- [1] Samir bouabdallah. *Design and Control of Quadrotors with Application to Autonomous Flying*. PhD thesis, Aboubekr Belkaid University, Tlemcen, Algeria, 2007.
- [2] Ian D. Cowling, James F. Whidborne, and Alastair K. Cooke. Optimal trajectory planning and LQR control for a quadrotor UAV. In *Control*, 2006.
- [3] Mark Cutler and Jonathan P. How. Actuator constrained trajectory generation and control for variable-pitch quadrotors. In *AIAA Guidance, Navigation, and Control Conference (GNC)*, Minneapolis, Minnesota, August 2012.
- [4] Li-Chun Lai, Chi-Ching Yang, and Chia-Ju Wu. Time-optimal control of a hovering quad-rotor helicopter. *Journal of Intelligent and Robotic Systems*, 45:115–135, 2006.
- [5] Daniel Mellinger and Vijay Kumar. Minimum snap trajectory generation and control for quadrotors. In *IEEE International Conference on Robotics and Automation (ICRA)*, May 2011.
- [6] Ivana Palunko, Rafael Fierro, and Patricio Cruz. Trajectory generation for swing-free maneuvers of a quadrotor with suspended payload: A dynamic programming approach. In *IEEE International Conference on Robotics and Automation (ICRA)*, May 2012.
- [7] Michael A. Patterson and Anil V. Rao. A general-purpose MATLAB toolbox for solving optimal control problems using the Radau pseudospectral method, 2013. www.gpops.org.

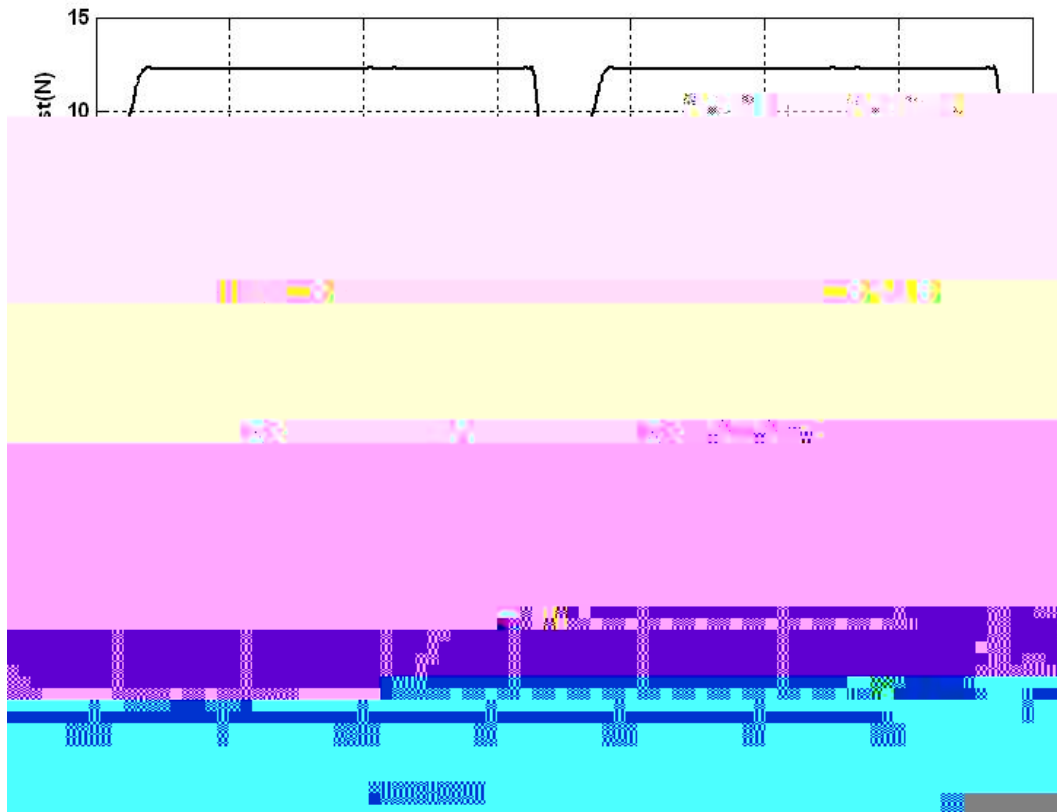


Figure 4: Optimal thrust force, roll and pitch angles vs time.