IMAV2016-29
http://www.imavs.org/pdf/imav.2016.29

Optimal Distributed Cooperative Control for Multiple Quadrotors: Heading Consensus Experiment with Paparazzi Autopilot and AR. Drone 2

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Abstract

An LQR control strategy for distributed coordination of multi-vehicle systems with linear dynamics is studied in this article. Using partitioning method, linear consensus algorithms for multi-vehicle systems with single-integrator dynamics is extended for high-order integral dynamics. Then the proposed high-order consensus protocol, is modelled as an optimal linear quadratic regulator (LQR) problem. Then, consensus protocol law is modified to achieve consensus merely on state variables while applying an optimal internal controller on higher states derivatives. A theorem is proposed to formulate the consensus strategy for this control problem and the stability analysis of multi-vehicle system is discussed using Lyapunov stability criterion. To illustrate the direct application of the proposed control theory and demonstrate the effectiveness of the proposed optimal high-order consensus protocol, linear second order heading dynamics of quadrotor is utilized to implement the consensus law on a real high-order dynamic MAV. Paparazzi open-source autopilot and the Parrot AR.Drone II commercial quadrotors are providing the software and hardware test bed for this experiment.

Keywords: Optimal Consensus, linear quadratic regulator (LQR), coordinated formation.

1. Introduction

Multi-Agent Systems are groups of vehicles capable of imitating human behavior as a team in the sense of coordination, cooperation and consensus. Agents in this venue are mobile robots in general operating on air, on the surface and below water. With emphasis to aerospace application, these agents are flying vehicles such as UAVs in formation, MAVs in cooperation and satellites in constellation. Intelligence of the vehicle in this regard can be defined as the ability of observing, decision making and directing its activities towards achieving an exclusive or inclusive goals.

There are tasks that sole vehicle cannot afford due to limitations in sensors, performance and cost. Therefore advantages such as scalability, flexibility, robustness, fault tolerance, easy maintenance, and less expensive schemes, can be achieved by having a group of vehicles work in collaboration with each other. As an important approach for these cooperative missions, cooperative control have received extensive attention [1] in recent control research areas. Formation control approaches

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like Swarming [2], distributed optimization [3], [4] and [5] potential field [6], Spatiotemporal planning like rendezvous [7], reconfiguration [8], coverage [9], cooperative tasking like assignment protocols [10] and consensus are among the famous categories in this field. The latter, i.e. consensus algorithms is a prevalent strategy in decentralized multi-vehicle cooperative control which have been studied widely in the literature. Consensus is based on exchanging the local neighbor-to-neighbor information between the vehicles, and it can be defined as: state-convergence of vehicles on a common value by negotiating with their neighbors or more conceptually, a distributed decision making process to agree on a value of interest. On the other hand, formation control can be proposed as maintaining desired spacing between the agents' position, velocity and higher order of derivatives relative to each other or relative to a reference. The combination of these two approaches lead to consensus formation control definition that can be proposed as: state-agreement of vehicles on particular values to provide a desired shape between vehicles states such as polygon.

Consensus problem is subject of research in both analysis and synthesis approaches. Typical problem specification in multi agent systems consensus algorithms include systems with time-delay communication links [11] and [12], time-dependent communication links [13], measurement noise [14], stochastic topology [15], analysis of graph Laplacian matrix theory [16], finite time convergence [17] and average seeking problems [18]. Regarding consensus protocol synthesis with respect to dynamics of agents, consensus laws are proposed for linear dynamics, including single, double and high-order systems; and nonlinear dynamics. In [19], consensus algorithms for single integrator kinematics was studied in different settings. Consensus for vehicles with double-integrator dynamics was studied in [20] wherein inspects the effect of relative state derivative availability to all agents or a subset of them. High-order integral dynamics was investigated in [21] which also presents the idea of high-order consensus with leader and the concept of model reference consensus problem. Regarding Consensus problem in nonlinear model context, the work in [22] is focused on Euler-Lagrangian nonlinear system for distributed coordination problem with directed topology consumption. Non-linear protocols have been proposed in [23] for optimal distributed consensus. Consensus reaching via distributed algorithms for general functions can be found in [24]. Optimal linear consensus algorithms for vehicles with single-integrator kinematics in a continuous-time setting from an LQR perspective have been studied in [19]. [25] has worked on optimal consensus protocol with the purpose of reaching minimum time convergence by deriving an optimal weighting matrix. Optimal average consensus problem was solved in [26], based on the optimal interaction graph approach. [27] designed distributed consensus algorithms which minimizes a collective objective function. In [28], the Linear Matrix Inequality (LMI) approach is addressed for optimization of consensus algorithm problems. Regarding optimal LQR control for input affine linear systems, [29] proposed a structured performance index to address the topology of the graph using an inverse optimality together with partial stability approach. [30] Proposes a framework for cooperative tracking control, considering a leader node as command generator. It also shows the duality of cooperative controller design and cooperative observer design in network systems. Taking external disturbance into account, in [31] a robust controller is proposed for a directed network of multiple agents with high-order integral dynamics, via defining an appropriate controlled output and a model transformation, which transfers the consensus performance into a normal $H\infty$ problem.

With emerging different control methodologies, optimality is one of the vital issues which is considered in the control science literature. Optimal considerations improve performance characteristics of control algorithms. In the case of linear differential dynamic systems with quadratic cost function, LQR theory is one of the most fundamental optimal control methods. Although, numerous linear-consensus algorithms investigate the optimality issues, mentioned above, optimal consensus for high-order dynamic systems is ignored. While in reality, a wide group of vehicles require second or higher order dynamic model. Dealing with high-order derivatives of states for LQR-based consensus formulation, makes the present work different from [19]. On the other hand, [32] deals with the same network system as of this paper, considering also the higher order terms in its proposed consensus protocol, but lacks in proposing an optimal controller formulation. Also unlike [29] that takes the inverse optimal approach where Laplacian matrix is known, we employ the direct approach in which Laplacian matrix in unknown and has to be defined via solution of the Riccati equation. In this sense, we comply with the prevalence in LQR problem venue in which Q and R matrices are design parameters and Laplacian matrix, being the optimal state/output feedback controller gain, is derived based on the solution of Algebraic Riccati equation (ARE).

Overally, the optimal LQR-based consensus algorithm contribution in the current research is twice. First, the work of Yongcan Cao and Wei Ren [19] is extended, in three aspects: 1) the high-order linear differential dynamics of vehicles are proposed in new partitioned form, 2) the optimal Laplacian matrix determination problem for high-order dynamics is modeled based on LQR theory and 3) the optimal LQR based consensus protocol for high-order dynamics is derived. Second, with the aim of preventing the consensus-reached value of states to vary, there should be consensus only on states as well as regulation on higher states derivatives, which leads to a novel optimal consensus with internal controller protocol for high-order dynamics. For this optimal control problem, a theorem with its stability proof based on Lyapunov function is formulated in this regard.

This paper is structured as follows: In Section 2, a brief definition of graph theory is presented. In Section 3, preliminaries of consensus problem for high-order dynamic system is described. Section 4 presents the optimal consensus and optimal internal controller formulation for the multi vehicle system via a theorem with stability analysis. Linear dynamic representation for quadrotor with focus on heading channel along with the special form of the proposed protocol for this channel are addressed in section 5. Finally, experimental results are given in Sections 6 while Section 7 contains conclusion and future work.

2. Graph theory

In general, the interaction graph between a group of n agents can be modeled by directed/undirected graph. Let $G(\nu,\varepsilon,\mathcal{A})$ be a directed graph, where $\nu=\{1,\dots,n\}$ is set of nodes, $\varepsilon \in \nu^2$ is an edge set of regular pairs of nodes and $\mathcal{A}=[a_{ij}] \in \mathbb{R}^{n\times n}$ is nonnegative adjacency matrix.

In a directed graph, edge $(i,j) \in \varepsilon$ notes that agent i can send its information to agent j, but it does not imply $(j,i) \in \varepsilon$. Whereas in undirected graph edge $(i,j) \in \varepsilon$ yields to $(j,i) \in \varepsilon$. The adjacency matrix represents the interconnection of adjacent nodes, that can be defined as $a_{ii}=0$, $a_{ij}>0$ if $(j,i) \in \nu$ and $i \neq j$. The nonsymmetrical Laplacian matrix $\mathcal{L}=[\ell_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $\ell_{ii}=\sum_{j\neq i} a_{ij}$ and $\ell_{ij}=-a_{ij}$, where $i \neq j$. The Laplacian matrix of undirected graph is symmetric positive semi-definite. In contrast, in directed graph $\mathcal{L}$ does not have this property. Let 1 be defined as $n\times 1$ column vector of all ones. With respect to the definition of $\mathcal{L}$, the Laplacian matrix has zero eigenvalue with an associated eigenvector 1. For undirected information exchange graph, $\mathcal{L}$ has simple zero eigenvalue if and only if the graph is connected. In case of directed graph, $\mathcal{L}$ has simple zero eigenvalue if and only if the directed information exchange graph has a directed spanning tree.

3. Dynamic system preliminaries

3.1 L^{th}-order dynamic system modeling

Assume n agents with L^{th}-order-integrator dynamics (n-by-L dynamics) is presented by:

$$\xi_i^{(L)}(t)= u_i(t), \qquad i \in \{1,\dots,n\}, \tag{1}$$

Where $\xi_i(t) \in \mathbb{R}$ is state parameter, $\xi_i^{(l)}(t)$ denotes L^{th} derivative of state, $u_i(t) \in \mathbb{R}$ is the control input and i^{th} presents the system number.

With respect to the existence of n agents that each of them is an L^{th}-order-integrator dynamic system, the matrix form statement of all agent dynamics prompts better conception. General matrix form of the system dynamics can be written by matrix partitioning technique. Consider

$$x^{(l)}(t)=\left[\,\xi_1^{(l)}(t),\dots,\xi_n^{(l)}(t)\right]^T, \qquad l \in \{1,\dots,L\}, \tag{2}$$

Therefore the matrix form is expressed as

$$\dot{\mathrm{X}}(t)=A\mathrm{X}(t)+BU(t), \tag{3}$$

With

$$\mathrm{X}(t)=\left[x(t),\dots,x^{(L-1)}(t)\right]^T, \quad U(t)=[u_1(t),\dots,u_n(t)]^T \tag{4}$$

And also

$$A=I_n\otimes\begin{bmatrix}0_{(L-1)\times 1} & I_{(L-1)\times(L-1)}\\ 0_{1\times 1} & 0_{1\times(L-1)}\end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} 0_{(\{L-1\}*n)\times n} & I_{(\{L-1\}*n)\times(\{L-1\}*n)} \\ 0_{n\times n} & 0_{n\times(\{L-1\}*n)} \end{bmatrix},$$

And

$$B = I_n \otimes \begin{bmatrix} 0_{(L-1)\times 1} \\ I_{1\times 1} \end{bmatrix} = \begin{bmatrix} 0_{(\{L-1\}*n)\times n} \\ I_{n\times n} \end{bmatrix}, \qquad (6)$$

Where $x^{(l)}(t) \in \mathbb{R}^{n\times 1}$, $\mathrm{X}(t) \in \mathbb{R}^{(L*n)\times 1}$, $\mathrm{U}(t) \in \mathbb{R}^{n\times 1}$ is vector of control inputs, $\mathrm{A} \in \mathbb{R}^{(L*n)\times(L*n)}$ and $\mathrm{B} \in \mathbb{R}^{(L*n)\times n}$ are constant coefficient matrices. I is identity matrix with dimensions according to the A and B matrices definition.

3.2 L^{th}-order dynamics consensus algorithms

To address the consensus formulation, we define the error state as:

$$\widetilde{\mathrm{X}}(\mathrm{t}) = \mathrm{X}(\mathrm{t}) - \mathrm{X_c} \qquad (7)$$

Where $\widetilde{\mathrm{X}}$ is the state error and $\mathrm{X_c}$ is the final consensus value of each agents' state and higher derivatives. According to equation (4) we have:

$$\widetilde{\mathrm{X}}(\mathrm{t}) = [\tilde{\mathrm{x}}, \dots, \tilde{\mathrm{x}}^{(\mathrm{L}-1)}] \qquad (8)$$

And according to equation (2) we have:

$$\tilde{\mathrm{x}}^{(\mathrm{l})} = \left[\left(\xi_1^{(\mathrm{l})}(\mathrm{t}) - \xi_{1\mathrm{c}}^{(\mathrm{l})}\right), \dots, \left(\xi_{\mathrm{n}}^{(\mathrm{l})}(\mathrm{t}) - \xi_{\mathrm{nc}}^{(\mathrm{l})}\right)\right]^{\mathrm{T}}, \mathrm{l} \in \{0,1,\dots,\mathrm{L}\} \qquad (9)$$

Where $\xi_{\mathrm{ic}}^{(\mathrm{l})}, \mathrm{i} \in \{1,\dots,\mathrm{n}\}, \mathrm{l} \in \{0,1,\dots,\mathrm{L}\}$ are the final consensus values for each agents' state i of order l and is reached when $\left\| \xi_{\mathrm{i}}^{(\mathrm{l})}(\mathrm{t}) - \xi_{\mathrm{j}}^{(\mathrm{l})}(\mathrm{t}) \right\|_{\mathrm{i}\neq\mathrm{j}} \to 0$.

Based on the above descriptions, the error dynamics or disagreement dynamics is:

$$\dot{\widetilde{\mathrm{X}}}(\mathrm{t}) = \mathrm{A}\widetilde{\mathrm{X}}(\mathrm{t}) + \mathrm{BU}(\mathrm{t}) \qquad (10)$$

The consensus algorithm of the L^{th}-order dynamic systems is expressed as [18]:

$$u_i(t) = -\sum_{j=1}^{n} a_{ij} \left[\sum_{k=0}^{L-1} (\gamma_{ij})_k \left(\xi_i^{(k)}(t) - \xi_j^{(k)}(t)\right)\right], \qquad (11)$$

Where $(\gamma_{ij})_0 = 1$, $(\gamma_{ij})_k > 0$, $k = 0, \dots, L-1$, $i, j = 1, \dots, n$, $\xi_i^{(k)}(t)$ presents k^{th} derivative of $\xi_i(t)$ that $\xi_i^{(0)}(t) = \xi_i(t)$, a_{ij} is (j, i) th entry of weighted adjacency matrix associated with interaction graph between the agents with $a_{ii} \triangleq 0$.

It can be noted that by applying protocol (11), each agent just needs its local neighbors' information. In case of n agents, the objective of (11) is declared, $\xi_i^{(k)}(t) \to \xi_j^{(k)}(t), k = \{0, \dots, L-1\}, \forall i \neq j$ as $t \to \infty$, which is the aim of consensus algorithms.

Interaction graph between a group of n agents is modeled by unidirectional graph $(\nu, \varepsilon, \mathcal{A})$. From the definition of $\mathcal{L}$ (Laplacian matrix) and also by substituting equation (11) in equation (3), general dynamics matrix form of n agents with L^{th}-order-integrator dynamic that are controlled by consensus algorithm can be defined as

$$\dot{\widetilde{\mathrm{X}}}(\mathrm{t}) = -\left(\begin{bmatrix} 0_{(\{\mathrm{L}-1\}*\mathrm{n})\times\mathrm{n}} & \mathrm{I}_{(\{\mathrm{L}-1\}*\mathrm{n})\times(\{\mathrm{L}-1\}*\mathrm{n})} \\ \mathcal{L}_{\mathrm{n}\times\mathrm{n}} & \Lambda_{\mathrm{n}\times(\{\mathrm{L}-1\}*\mathrm{n})} \end{bmatrix}\right)\widetilde{\mathrm{X}}(\mathrm{t}), \qquad (12)$$

With

$$\Lambda = [\,\eta_1\mathcal{L}, \dots, \eta_{L-1}\mathcal{L}\,], \qquad (13)$$

It is also worthy noticing that the pair $(\mathrm{a_{ii}}, \gamma_{\mathrm{ij}})$ in (11), plays the same role as the pair $(\mathcal{L}, \eta)$ in (16), while regarding to the definition of Laplacian matrix $\mathcal{L}$ in section 2 , as the Laplacian matrix $\mathcal{L}$ is associated with $\mathcal{A}$, the η is associated with Γ, Where $\Gamma = [\gamma_{\mathrm{ij}}] \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. It is clear that equation (16) is a linear differential equation. Considering the definition of $\mathrm{X}(\mathrm{t})$ in equation (4), equation (16) can be expressed as follows: The last $n-th$ entry of the $\dot{\widetilde{\mathrm{X}}}(\mathrm{t})$ matrix, corresponds to the control inputs as are defined in equation (4) and can be written as following matrix form equation:

$$U(t) = -\mathcal{K}\,X(t)\,,$$
where
$$\mathcal{K} = [\mathcal{L} \quad \eta_1\mathcal{L} \quad \dots \quad \eta_{L-1}\mathcal{L}]\,, \tag{14}$$

Therefore, the consensus protocol of n agents with L^{th}-order dynamics based on Laplacian matrix of interaction graph between the agents i.e. equation (14), can be presented in algebraic form by:

$$U(t) = -\sum_{k=0}^{L-1} \eta_k \mathcal{L} x^{(k)}(t), \tag{15}$$

Where $\boldsymbol{\eta_0} = \boldsymbol{I_{n\times n}}, \boldsymbol{\eta_k} \in \mathbb{R}^{\boldsymbol{n\times n}}$ is unknown constant coefficients matrix that are associated with $\boldsymbol{k^{th}}$ derivative of $\boldsymbol{x(t)}$ and $\boldsymbol{k = 0, \dots, L-1}$.

4. Optimal cooperative controllers

4.1 LQR-consensus controller

From equation (3), it is clear that general matrix form of n agents with L^{th}-order-integrator dynamics is modeled as a system with linear differential equation therefore similar to the optimal-control problems, the consensus cost function can be proposed as

$$J = \int_0^\infty \{\sum_{i=1}^{n}\sum_{j=1}^{n-1} \left(\sum_{k=0}^{L-1}\sum_{\acute{k}=0}^{k-1} (q_{ij})_{k\acute{k}} \left(\xi_i^{(k)}(t) - \xi_j^{(k)}(t)\right)^2\right) + \sum_{i=1}^{n} r_i\, u_i^2(t)\}\, dt, \tag{16}$$

Where $(q_{\mathrm{ij}})_{k\acute{k}} \geq 0 \in \mathbb{R}$ and $\mathrm{r_i} > 0 \in \mathbb{R}$ can be chosen freely; therefore, J in equation (16), can be called as an interaction-free cost function. Therefore the optimization problem can be proposed as

$$min_{a_{ij},\gamma_k}\ J \text{, subject to (3) and (15),} \tag{17}$$

According to protocol (15), deriving the optimal consensus algorithms is equivalent to deriving optimal Laplacian matrix $\mathcal{L}$ and optimal constant coefficients matrix η_k. to derive an optimal solution, continuous-time setting of linear quadratic regulator (LQR) algorithm is hereby utilized. Optimal control problem (17) subject to linear system which is defined as equation (3) and interaction related cost function (16) can be written as

$$min_{a_{ij},\gamma_k}\ J = \int_0^\infty [X^T(t)\,Q\,X(t) + U^T(t)\,R\,U(t)]\ dt, \tag{18}$$
$$\text{Subject to } \dot{X}(t) = AX(t) + BU(t),$$
$$U(t) = -\sum_{k=0}^{L-1} \eta_k \mathcal{L} x^{(k)}(t),$$
$$\text{Where, } X(t) = \left[x(t), \dots, x^{(L-1)}(t)\right]^T,$$

Where $X(t), U(t), A, B$ are defined in (3), $x^{(k)}$ is defined in (2), $Q = [q_{ij}] \in \mathbb{R}^{(L*n)\times(L*n)}$ is a symmetric positive semi-definite (PSD) design matrix associated with the multi agent system. Each diagonal element $q_{ii} \in \mathbb{R}^{n\times n}$ represents designer's desired weighted interaction on the agents' states and its higher derivatives of the same type, and the non-diagonal terms accounts for the cross-type weighting between states and derivatives i.e. q_{ij}, $i \neq j$ are transversal weighting design matrices. In its simplest form, we can ignore cross-type weighting and also assign a sole design matrix q_{11} on the states and expand it for higher derivatives. . q_{11} is chosen as a symmetric Laplacian matrix associated with a connected graph. In this case, Q can be defined in the Kronecker form of equation (21). $\mathrm{R} \in \mathbb{R}^{n\times n}$ is a positive-definite diagonal matrix which r_i is ith diagonal entry and $\mathcal{L}, \eta_k$ are optimal Laplacian and optimal constant coefficients matrix that is defined in (15) and are going to be derived. This problem's solution can be explained by following steps.

Step 1: With the aim of deriving the general equation of optimal Laplacian $\mathcal{L}$ and constant coefficients matrix η_k for n agents with L^{th}-order-integrator dynamic, constant coefficients matrix in equation (18) should be written in partitioned matrix form. With respect to equations (5) and (6), it can be proposed as

$$A = \begin{bmatrix} 0 & I & 0 & 0 & \dots & 0 \\ 0 & 0 & I & 0 & \dots & 0 \\ 0 & 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{19}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}, \tag{20}$$

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1L} \\ q_{21} & q_{22} & \cdots & q_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ q_{L1} & q_{L2} & \cdots & q_{LL} \end{bmatrix} = (I_L \otimes q_{11}), \tag{21}$$

Where A, B are defined in equations (5) and (6), respectively and Q is defined in equation (18). Each of the matrices entries denotes an n-by-n matrix (i.e., a matrix with n rows and n columns). Whereas, R is defined in equation (18) as an n-by-n matrix, does not need partitioning. 0 and I donate null and identity matrices, respectively.

Step 2: In this step, the optimal problem (18) is alternated to standard LQR problem by considering simple form of controller input and then analytical solution for n-by-L dynamics (3), is derived. The simplified problem can be expressed as

$$\begin{aligned} &min_{U(t)}\ J \\ &\text{Subject to: } \dot{X}(t) = AX(t) + BU(t), \\ &\qquad\qquad U(t) = -\mathcal{K}X(t), \\ &\text{Where, } \mathcal{K} = R^{-1}B^T P \end{aligned} \tag{22}$$

Where J is defined in equation (18), $\mathcal{K} \in \mathbb{R}^{n\times(L*n)}$ is optimal control command's coefficient matrix and $P = [p_{ij}] \in \mathbb{R}^{(L*n)\times(L*n)}$ satisfies the continuous-time algebraic Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0_{(L*n)\times(L*n)} \tag{23}$$

Where each of the matrix entries p_{ij}, denote an n-by-n matrix. For solving the simplified optimal problem (22), first, by applying equations (19), (20), (21) and (24) with respect to R is defined in (18), equation (23) can be written in matrix form

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1L} \\ p_{21} & p_{22} & \cdots & p_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ p_{L1} & p_{L2} & \cdots & p_{LL} \end{bmatrix}, \tag{24}$$

$$\begin{bmatrix} \Pi_{1_{n\times n}} & \Pi_{2_{n\times(\{L-1\}*n)}} \\ \Pi_{3_{(\{L-1\}*n)\times n}} & \Pi_{4_{(\{L-1\}*n)\times(\{L-1\}*n)}} \end{bmatrix} = 0_{(L*n)\times(L*n)}, \tag{25}$$

Where, $\Pi_i\ i = \{1, \dots, 4\}$ is proposed in equation (22). For n agents with L^{th}-order-integrator dynamic with specific interaction graph among agents G, L and Q are known, and R is noted weighted control coefficient matrix, which is supposed to be known for all direct optimal problem. Therefore, by solving matrix system of equation (25), $P = [p_{ij}]$ can be computed by (26).

On the other hand, applying equations (20) and (24) with assuming R as presented by (18), simplified control command matrix equation (22), can be given by following partitioned matrix form equation

$$\begin{aligned} &\Pi_1 = -R^{-1} p_{1L} p_{L1} + q_{11}, \\ &\Pi_2 = [p_{11} - R^{-1} p_{1L} p_{L2} + q_{12}, \dots, p_{(L-1)1} - R^{-1} p_{1L} p_{LL} + q_{1L}], \\ &\Pi_3 = [p_{11} - R^{-1} p_{2L} p_{L1} + q_{21}, \dots, p_{(L-1)1} - R^{-1} p_{LL} p_{L1} + q_{L1}], \\ &\Pi_4 = \begin{bmatrix} p_{12} + p_{12} - R^{-1} p_{2L} p_{L2} + q_{22} & \cdots & p_{1L} + p_{(L-1)2} - R^{-1} p_{2L} p_{LL} + q_{2L} \\ \vdots & \ddots & \vdots \\ p_{1L} + p_{(L-1)2} - R^{-1} p_{LL} p_{L2} + q_{L2} & \cdots & p_{(L-1)L} + p_{(L-1)L} - R^{-1} p_{LL} p_{LL}\ q_{LL} \end{bmatrix}, \end{aligned} \tag{26}$$

$$U(t) = -[R^{-1}\wp_{L1}, \dots, \; R^{-1}\wp_{LL}]\,\mathrm{X} \qquad (27)$$

This equation represents the optimal LQR-based control command which is the foundation of general form of optimal consensus control protocol.

4.2 LQR- state Consensus with internal control on derivatives

One popular approach in consensus problems is to reach an agreement only on the state such as position and orientation, forcing and keeping higher orders of that state zero[31-32]. For example in the case that $\xi_i(t)$ denotes the position of i th agent, ${\xi_i}^{(1)}(t), {\xi_i}^{(2)}(t), \dots, {\xi_i}^{(L)}(t)$, denote velocity, acceleration and L^{th} derivative of position state of ith agent respectively. If the condition is to hold the consensus-reached position, thus the velocities and other derivatives of position must be driven to zero. It means that, applying consensus protocol on states and exerting an internal controller on higher order derivatives. Therefore, the equation (11) should be rewritten as

$$\begin{aligned} {u_i}'(t) = &-\sum_{j=1}^{n} a_{ij}\{\xi_i(t) - \xi_j(t)\} \\ &-\sum_{j=1}^{n} a_{ij}\{\sum_{k=1}^{L-1} (\gamma_{ij})_k\, \xi_i^{(k)}(t)\}, \\ &i \in \{1, \dots, n\} \end{aligned} \qquad (28)$$

With regard to the relation between the adjacency matrix and degree matrix arguments for undirected graph of n agents, i.e. $\sum_{j=1}^{n} a_{ij} = d_i, i \in \{1, \dots, n\}$ [33], where d_i denotes the degree of ith agent, and also, as $\sum_{j=1}^{n} a_{ij}$ is independent of the counter of the second sigma, it follows that

$$\begin{aligned} {u_i}'(t) = &-\sum_{j=1}^{n} a_{ij}\{\xi_i(t) - \xi_j(t)\} \\ &-\sum_{k=1}^{L-1} d_i.(\gamma_{ij})_k\, \xi_i^{(k)}(t), \\ &i \in \{1, \dots, n\} \end{aligned} \qquad (29)$$

Rewriting this equation in the matrix form, like what has been done for equation (15) yields to

$$U'(t) = -\mathcal{L}\tilde{x}(t) - \sum_{k=1}^{L-1} \eth_k D\, x^{(k)}(t), \qquad (30)$$

Where $D = diag\{d_i\} \in \mathbb{R}^{n\times n}$ is the degree matrix of interaction graph between agents. Considering $\mathcal{L} = D - \mathcal{A}$ [33], the equation can be expressed as

$$U'(t) = \mathcal{A}x(t) - \sum_{k=1}^{L-1} \delta_k D \tilde{x}^{(k)}(t), \qquad (31)$$

$\delta_k \in \mathbb{R}^{n\times n}$ where $\delta_k D$ is equivalent but not equal to $\eta_k \mathcal{L}$ for $k = \{1, \dots, L-1\}$ and $\Delta = [\,\delta_1 D, \dots, \delta_{L-1} D\,]$ is equivalent to Λ in (13). By defining $W \in \mathbb{R}^{n\times(L*n)}$ where, $w_{ij} = a_{ij}$ for $i \; and \; j \in \{1, \dots, n\}$ and $w_{ij} = 0$ for $j > n$, the equation (31) is proposed as

$$U'(t) = W\tilde{X}(t) - \sum_{k=1}^{L-1} \delta_k D \tilde{x}^{(k)}(t), \qquad (32)$$

As of step 2 presentation in the part 4.1, in order to reach the optimal consensus protocol with optimal internal controller, one should equalize the consensus protocol (15) and optimal control command (22). Hence, in the same stream, the above equation will be as follows

$$\begin{aligned} U'(t) &= W\tilde{\mathrm{X}}(t) - \mathcal{K}\tilde{X}(t) \\ \mathcal{K} &= [D \quad \delta_1 D \quad \dots \quad \delta_{L-1} D], \end{aligned} \qquad (33)$$

At last, according to the definition of optimal control command in equation (22), the optimal consensus with internal controller protocol illustrated as

$$U'(t) = W\tilde{\mathrm{X}}(t) - U(t), \qquad (34)$$

Where U is optimal internal control command for state derivatives while the consensus attributes lay in $W\mathrm{X}(t)$ term. While just control command changes, the new optimal control problem subject to the linear system (3) is derived by substituting (33) in equation (18), as

$$min_{a_{ij},\gamma_k}\; J = \int_0^\infty \left[\mathrm{X}^T(t)\, Q\tilde{\mathrm{X}}(t) + {U'}^T(t) R\, U'(t)\right] dt, \qquad (35)$$

$$\text{Subject to: } \dot{\tilde{\mathrm{X}}}(t) = A\tilde{\mathrm{X}}(t) + BU'(t),$$
$$U'(t) = W\tilde{\mathrm{X}}(t) - \mathcal{K}\tilde{\mathrm{X}}(t),$$
$$\text{Where, } \tilde{\mathrm{X}}(t) = \left[\tilde{x}(t), \dots, \tilde{x}^{(L-1)}(t)\right]^T,$$

In order to write the new optimal problem in a comparable form with equation (18), i.e. (defining the new problem with use of parameters of equation (18)), the equation (34) should be substitute in equation (35)

$$J = \int_0^\infty [\tilde{X}^T(t) Q\, \tilde{X}(t) + (W\tilde{X}(t) - U(t))^T(t) R\, (W\tilde{X}(t) - U(t))(t)]\; dt, \tag{36}$$

$$J = \int_0^\infty [\tilde{X}^T(t)\, Q\, \tilde{X}(t) + X^T(t)\, W^T R W\, \tilde{X}(t) + U^T(t) R W\, \tilde{X}(t) + \tilde{X}^T(t)\, W^T R\, U(t) + U^T(t) R\, U(t)]\; dt,$$

By defining $Q' = Q + W^T R W$ and $\mathcal{K}' = R^{-1}(B^T P - R_1)$, where, $R_1 = R\,W$, and assuming that $U^T(t) R W\, X(t)$ is a symmetric matrix, the new optimal problem can expressed as

$$min_{a_{ij},\gamma_k}\; J = \int_0^\infty [\tilde{X}^T(t)\, Q'\, \tilde{X}(t) + 2\tilde{X}^T(t)\, R_1\, U(t) + U^T(t) R\, U(t)]\; dt, \tag{37}$$

$$\text{Subject to: } \dot{\tilde{X}}(t) = A\tilde{X}(t) + BU(t),$$
$$U(t) = -\mathcal{K}'\tilde{X}(t),$$
$$\text{Where, } \tilde{X}(t) = \left[\tilde{x}(t), \ldots, \tilde{x}^{(L-1)}(t)\right]^T,$$
$$Q' = Q + W^T R W,$$
$$\mathcal{K}' = R^{-1}(B^T P - R_1),$$
$$\text{Where, } R_1 = R\,W$$

Before moving on, the following lemma succeeded by a theorem are introduced to prove asymptotic stability of the proposed optimal control command.

Lemma 1. Consider the dynamical system

$$\dot{\tilde{X}}(t) = f\left(\tilde{X}(t), U(t)\right), \quad \tilde{X}(0) = \tilde{X}_0, \quad t \geq 0, \tag{38}$$

With $f(0,0) = 0$ and a cost function proposed by

$$J(\tilde{X}_0, U(0)) = \int_0^\infty \mathrm{T}\left(\tilde{X}(t), U(t)\right)\, dt, \tag{39}$$

Where the state feedback controller $U(t) = \phi\left(\tilde{X}(t)\right)$ minimizes the cost function and the $U(0)$ is an admissible control which can be defined as

$$U(0) = \phi\left(\tilde{X}(0)\right), \tag{40}$$

Let $D \subseteq \mathbb{R}^n$ be an open set and $\Omega \subseteq \mathbb{R}^L$, assume $0 \in D$, let $f: D \to \mathbb{R}^n$, $\phi: D \to \Omega$, and $\mathrm{T}: D \to \mathbb{R}$. Assume there is a continuously differentiable function $V: D \to \mathbb{R}$ such that

$$V(0) = 0, \tag{41}$$

$$V(\tilde{X}) > 0, \quad \tilde{X} \in D, \quad \tilde{X} \neq 0, \tag{42}$$

$$V'(\tilde{X}) f\left(X(t), \phi\left(\tilde{X}(t)\right)\right) < 0, \quad \tilde{X} \in D, \tilde{X} \neq 0, \tag{43}$$

$$\mathrm{T}\left(\tilde{X}(t), \phi(X(t))\right) + V'(\tilde{X}) f\left(\tilde{X}(t), \phi(\tilde{X}(t))\right) = 0, \tilde{X} \in D, \tag{44}$$

Then the zero solution $\tilde{X}(t) \equiv 0$ of closed-loop system

$$\dot{\tilde{X}}(t) = f\left(\tilde{X}(t), \phi\left(\tilde{X}(t)\right)\right), \quad \tilde{X}(0) = \tilde{X}_0, \quad t \geq 0 \tag{45}$$

Is locally asymptotically stable, and

$$J\left(\tilde{X}_0, U(0)\right) = V(\tilde{X}_0), \quad \tilde{X}_0 \in D, \tag{46}$$

Finally, if $D = \mathbb{R}^n$, $\Omega = \mathbb{R}^L$, and

$$V(\tilde{X}) \to \infty \quad as \quad \|\tilde{X}\| \to \infty, \tag{47}$$

Then the zero solution $X(t) \equiv 0$ of closed-loop system is globally asymptotically stable.

Proof: omitted. Refer to [34].

According to the multi-agent system dynamics representation of equations (1) to (6), and also definitions in (37), this study is confined to a specific form of system (22), where $U(t) = -\mathcal{K}'\tilde{X}(t)$ and $\mathcal{K}' = R^{-1}(B^T P - R_1)$ and the formal optimal regulator equation $\mathcal{K} = R^{-1}B^T P$ in which by substituting the B as defined in equation (6), is $\mathcal{K} = R^{-1}[\wp_{L1}, \ldots, \wp_{LL}]$, thus we have $\mathcal{K}' = R^{-1}[(\wp_{L1} - R_1), \wp_{L2}, \ldots, \wp_{LL}]$.

Step 3: As the last step, the optimal Laplacian and optimal constant coefficients matrix should be derived. It can be done by according the extracted control law matrix $\mathcal{K}'$ mentioned above, to the control command matrix form of equation (33). First, by considering $\mathcal{K}'$ the command law U' can be rewritten

$$U'(t) = -\{(R^{-1}\mathscr{p}_{L1} - \mathcal{A})\tilde{x}(t) + R^{-1}\mathscr{p}_{L2}\dot{\tilde{x}}(t) \dots + R^{-1}\mathscr{p}_{LL}\tilde{x}^{(L-1)}(t)\}, \quad (48)$$

And expansion form of equation (31) is given by

$$U'(t) = -\{\delta_0 D\tilde{x}(t) + \dots + \delta_{(L-1)}D\tilde{x}^{(L-1)}(t)\}, \quad (49)$$

By matching equations (48) and (49), the optimal Laplacian matrix, $\mathcal{L}$ and optimal constant coefficients matrix, $\eta_{(k)}$, where $k = 1, \dots, L-1$ are derived as

$$D = R^{-1}\mathscr{p}_{L1},$$

And,

$$\delta_0 = \mathcal{L}D^{-1} \quad (50)$$
$$\delta_{(k)} = R^{-1}\mathscr{p}_{L(k+1)}D^{-1}, \qquad k \in \{1, \dots, L-1\},$$

Theorem 1. The matrix $\mathcal{L}$ in Equation (50) is a Laplacian matrix and is equal to $\sqrt{\mathrm{R}^{-1}\, q_{11}}$.

Proof: According to the first partition of Equation (26), we have $-\mathrm{R}^{-1}\mathscr{p}_{1\mathrm{L}}\mathscr{p}_{\mathrm{L}1} + q_{11} = 0$, on the other side, with regard to LQR problem setting, we know that P is symmetric matrix, meaning that $\mathscr{p}_{1\mathrm{L}} = \mathscr{p}_{\mathrm{L}1}$. Therefore, it is followed that $\mathscr{p}_{\mathrm{L}1} = \sqrt{\mathrm{R}\, q_{11}}$.

The proof is finalized by substituting $\mathscr{p}_{\mathrm{L}1}$ into Equation (50), which is yield to $\mathcal{L} = \mathrm{R}^{-1}\mathscr{p}_{\mathrm{L}1} = \sqrt{\mathrm{R}^{-1}\, q_{11}}$. So by Assigning q_{11} as a symmetric Laplacian matrix with a simple zero eigenvalue associated with a connected graph, $\mathrm{R}^{-1}\, q_{11}$ is a (non-symmetric) Laplacian matrix with a simple zero eigenvalue, since R is a positive definite diagonal matrix.

Therefore, the partitioned matrix form of optimal consensus protocol for n agents with L^{th} -order-integrator dynamic is derived as

$$U(t) = -\sum_{k=0}^{L-1}\delta_k D\tilde{x}^{(k)}(t), \quad (51)$$

$U(t)$ and $P = [\mathscr{p}_{ij}]$ are defined in equations (4) and (20) , respectively. D and δ_k are n-by-n matrices.

With the aim of deriving the optimal consensus algorithm of n agents with L^{th} -order-integrator dynamic, the optimal consensus protocol (2) should be rewritten in basic consensus algorithm form (11). According to the presented procedure of deriving equations (12-15) by use of equations (3) and (11), the optimal consensus algorithm is given by

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}\{\xi_i(t) - \xi_j(t)\} - \sum_{k=1}^{L-1} d_i.(\gamma_{ij})_k\, \xi_i^{(k)}(t), \quad i \in \{1, \dots, n\},$$

Where,

$$\begin{cases} \Gamma_k = [\gamma_{ij}]_k \\ \Gamma_0 = 1_{n\times n} & k = 0\,, \\ \Gamma_k = (R^{-1}\mathscr{p}_{L(k+1)})\mathcal{A}^{-1} & k \neq 0 \end{cases} \quad (52)$$

$$\mathcal{A} = [a_{ij}], \quad where \begin{cases} a_{ij} = -\ell_{ij} & i \neq j \\ a_{ij} = 0 & i = j \end{cases},$$

$$D = [d_i], \qquad i \in \{1, \dots, n\},$$

Note that from the equality of (48) and (49), we have $R^{-1}\mathscr{p}_{L(k+1)} = \delta_k D$, and we already know that D is a diagonal matrix, thus $R^{-1}\mathscr{p}_{L(k+1)}, k = \{0, \dots, L-1\}$ are all degree matrices.

Based on the above lemma and the afterward explanations, we propose the following theorem complying the aforementioned control protocols for linear system and quadratic performance index.

Theorem 1. Consider the linear dynamical system

$$\dot{\tilde{\mathrm{X}}}(t) = A\tilde{\mathrm{X}}(t) + BU(t), \quad \tilde{\mathrm{X}}(0) = \tilde{\mathrm{X}}_0, \quad t \geq 0, \quad y(t) = -\mathcal{K}'(\tilde{\mathrm{X}}(t)), \quad (53)$$

With Quadratic performance function

$$J(\tilde{\mathrm{X}}_0, U(0)) = \int_0^\infty (\tilde{\mathrm{X}}^T(t)\, Q'\, \tilde{\mathrm{X}}(t) + 2\tilde{\mathrm{X}}^T(t)\, R_1\, U(t) + U^T(t) R\, U(t))dt, \quad (54)$$

Where $U(t) = -\mathcal{K}'\tilde{\mathrm{X}}(t)$ is the state feedback controller, which the optimal regulator is $\mathcal{K}' = R^{-1}(B^T P - R_1)$, and $U(0)$ is an admissible control.

Then, $V = \tilde{\mathrm{X}}^T P\tilde{\mathrm{X}}$ can be proposed as a candidate Lyapunov function, and the zero solution of closed-loop system

$$\dot{\tilde{\mathrm{X}}}(t) = (A - B\mathcal{K}')\tilde{\mathrm{X}}(t), \quad \tilde{\mathrm{X}}(0) = \tilde{\mathrm{X}}_0, \quad t \geq 0, \quad (55)$$

Is globally asymptotically stable.

Proof: By using lemma 1, if the candidate Lyapunov function $V = \tilde{X}^T P \tilde{X}$ can satisfies all required conditions, hence, for all $\tilde{X} \in D$ this theorem is applicable .

The time derivative of candidate Lyapunov function yields

$$\dot{V}(\mathrm{X(t)}) = \dot{\mathrm{X}}(t)^T P\, \mathrm{X(t)} + \mathrm{X(t)}^T P\, \dot{\mathrm{X}}(t), \qquad (56)$$

By substituting the closed-loop system equation (55) and optimal regulator $\mathcal{K}' = R^{-1}(B^T P - R_1)$

$$\begin{aligned} \dot{V}(\mathrm{X(t)}) = {} & [(A - B\mathcal{K}')\mathrm{X(t)}]^T P\, \mathrm{X(t)} \\ & + \mathrm{X(t)}^T P\, [(A - B\mathcal{K}')\mathrm{X(t)}], \qquad (57) \\ = {} & \mathrm{X(t)}^T [(A - B\{R^{-1}(B^T P - R_1)\})^T P \\ & + P(A - B\{R^{-1}(B^T P - R_1)\})]\mathrm{X(t)}, \\ = {} & \mathrm{X(t)}^T [PA + A^T P - 2PBR^{-1}B^T P \\ & + R_1 R^{-1} B^T P + PBR^{-1} R_1]\mathrm{X(t)}, \end{aligned}$$

Due to the algebraic Riccati equation which is correspond to quadratic performance function (54) [35], there exist the following equality

$$\begin{aligned} 0 = {} & PA + A^T P - PBR^{-1}B^T P - R_1 R^{-1} R_1 \qquad (58) \\ & + R_1 R^{-1} B^T P + PBR^{-1} R_1 + Q', \end{aligned}$$

The time derivative of candidate Lyapunov function is henceforth represented as

$$\begin{aligned} \dot{V}\left(\tilde{X}(\mathrm{t})\right) = {} & \tilde{X}(\mathrm{t})^T [-Q' + R_1 R^{-1} R_1 \qquad (59) \\ & - PBR^{-1}B^T P]\tilde{X}(\mathrm{t}), \end{aligned}$$

Recalling the definitions $Q' = Q + W^T R W$ and $R_1 = R\, W$, one has

$$\dot{V}\big(\mathrm{X(t)}\big) = \tilde{X}(\mathrm{t})^T [-Q - PBR^{-1}B^T P]\tilde{X}(\mathrm{t}), \qquad (60)$$

Reaffirmed that, P is positive-definite matrix which is defined in equation (22), hence for $R > 0$ and $R^{-1} > 0$, $PBR^{-1}B^T P$ is positive-definite matrix too, i.e. $PBR^{-1}B^T P > 0$. Also Q is positive-semi-definite matrix, i.e. $Q \geq 0$ which is defined in equation (18), hence, $PBR^{-1}B^T P + Q > 0$ and $\dot{V} < 0$. Furthermore, as the Frechet derivative $(\acute{\cdots})$ is mentioned by equation (44), hence, $\acute{V}\big(\tilde{X}(\mathrm{t})\big)\, f\big(\tilde{X}(t), U(t)\big) < 0$ for all $\tilde{X} \in D$.

On the other hand, substituting the state feedback controller function in Quadratic performance function (54) yields

$$\begin{aligned} J\big(\mathrm{X}_0, U(0)\big) = {} & \int_0^\infty \big(\tilde{X}^T(t)\, Q'\, \tilde{X}(t) + 2\tilde{X}^T(t)\, R_1\, U(t) \qquad (61) \\ & + U^T(t) R\, U(t)\big) dt, \end{aligned}$$

$$\begin{aligned} J\left(\tilde{X}_0, U(0)\right) = {} & \int_0^\infty \Big(\tilde{X}(\mathrm{t})^T Q'\, \tilde{X}(\mathrm{t}) + \\ & 2\tilde{X}^T(t)\, R_1 \left(-R^{-1}(B^T P - R_1)\tilde{X}(\mathrm{t})\right) + \\ & \left(-R^{-1}(B^T P - R_1)\tilde{X}(\mathrm{t})\right)^T R \Big(-R^{-1}(B^T P - \\ & R_1)\tilde{X}(\mathrm{t})\Big)\Big) dt, \\ = {} & \int_0^\infty \tilde{X}(\mathrm{t})^T [Q' - R_1 R^{-1} R_1 \\ & + PBR^{-1}B^T P]\tilde{X}(\mathrm{t})\, dt, \end{aligned}$$

And hence, by (59)

$$\begin{aligned} J\left(\tilde{X}_0, U(0)\right) = {} & \int_0^\infty \left(-\dot{V}\left(\tilde{X}(\mathrm{t})\right)\right) dt \\ = {} & \underbrace{-V\left(\tilde{X}(\mathrm{t} \to \infty)\right)}_{\to 0} + V\big(\tilde{X}_0\big) \qquad (62) \\ = {} & V\big(\tilde{X}_0\big), \qquad for\ all\ \tilde{X}_0 \in D, \end{aligned}$$

Also, by (62), (39) and (61) it can be concluded that

$$\begin{aligned} \mathrm{T}\left(\tilde{X}(t), \phi\big(\tilde{X}(\mathrm{t})\big)\right) = {} & -\dot{V}\left(\tilde{X}(\mathrm{t})\right) \qquad (63) \\ = {} & \acute{V}\big(\tilde{X}\big) f\left(\tilde{X}(t), \phi\big(\tilde{X}(\mathrm{t})\big)\right), \qquad \tilde{X} \in D, \end{aligned}$$

Then, the last necessary condition is satisfied too, i.e. $\mathrm{T}\big(\tilde{X}(t), \phi(\tilde{X}(\mathrm{t}))\big) + \acute{V}(\tilde{X}) f\big(\tilde{X}(t), \phi(\tilde{X}(\mathrm{t}))\big) = 0$, for all $\tilde{X} \in D$. Therefore, the zero solution of closed-loop system (55) is globally asymptotically stable.

The next session discusses the formation strategy but before moving on, one must notice that for a multi agent system with a MIMO-dynamical system for each agent, by defining $\xi|_\mu$ and $u|_\mu$, $\mu \in \{1, \dots, m\}$, the proposed control law (52) can be rewritten as:

$$u_i(t)|_\mu = -\sum_{j=1}^{n} a_{ij} \left[\sum_{k=0}^{L-1} (\gamma_{ij})_k \left(\xi_i^{(k)}(t)|_\mu - \xi_j^{(k)}(t)|_\mu \right) \right], \quad i \in \{1, \dots, n\}, \mu \in \{1, \dots, m\} \tag{64}$$

Where $u_i(t)|_\mu$ and $\xi_i(t)|_\mu$ are the $\mu-$th entries of state and input vectors ξ and u, respectively.

5. Quadrotor Linearized Heading Dynamics

Proposing a simplified model for control design where the aerodynamic, gyroscopic and Coriolis forces and moments are negligible, and assuming small changes for the Euler angles, meaning that $[\varphi, \theta, \psi] \simeq [p, q, r]$ and having linearized the equations of motion around $\varphi_0, \theta_0, \psi_0, u_{1_0}$, we obtain:

$$\begin{cases} \ddot{x} = \theta \\ \ddot{y} = \varphi \\ \ddot{z} = u_1 \\ \ddot{\varphi} = u_2 \\ \ddot{\theta} = u_3 \\ \ddot{\psi} = u_4 \end{cases} \tag{65}$$

We define the state vector $\xi = [x, y, z, \varphi, \theta, \psi]$ and input vector $u = [u_1, u_2, u_3, u_4]$. In the above equations, the input force and torque signals are normalized per mass and moment of inertia for quadrotor, respectively.

Considering the second order integral dynamics for heading (ψ), the optimal consensus algorithm for heading state and its first derivative is

$$u_{4,i} = -\sum_{j=1}^{n} a_{ij} \left[(\psi_i - \psi_j) + \gamma_{ij}\, \dot{\psi}_i \right], i \in \{1, \dots, n\} \tag{66}$$

Since all simulation results, not only for heading state but also for three dimensional position consensus formation is delivered in [36], the reader is referred to that paper for simulations details, thus in this paper we focus on the experimental aspects of this consensus algorithm.

6. Experiment

Paparazzi is an open-source autopilot software that have attracted a lot of attention from academics in the field of drone programming all over the world. Parrot AR.Drone II, on the other side is a ubiquitous commercial quadrotor with open access micro-processor. Paparazzi software can be setup on the AR.Drone II's computer [37]. We employ this combination of hardware-software to implement the consensus protocol (66) in a real-world experiment. WiFi connection between the drones can provide instantaneous data exchange viz. heading angles of the drones amongst each other. Each drone calculates the u_4 command separate on its processer but alternatively command generating for all agents can be performed on a ground computer and uplinked to the relevant drone. Thus each drone must downlink its data to the ground computer and receive the u_4 command back from it. In this sense, the consensus algorithm still has its distributed nature, but the data manipulation and command calculation is implemented in a centralized manner.

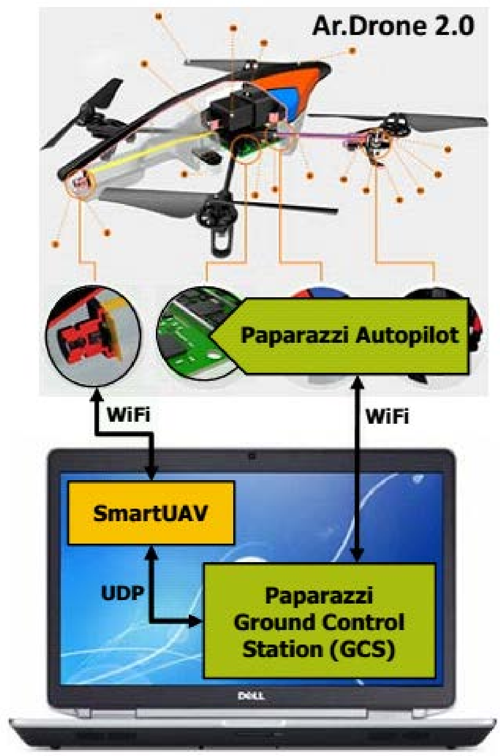

Figure 1. Test bed system architecture [38]

Now, consider a network of 3 quadrotors which are stabilized in hover mode while their heading channels are prone to follow consensus algorithm described by equation (66). Optimal designing parameters Q and R are proposed as follows

$$\mathrm{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{I_L} \otimes \mathrm{r}_1,$$

$$\mathrm{Q} = \mathrm{I_L} \otimes q_{11},$$

Where, (67)

$$\mathrm{r}_1 = 1,$$

$$q_{11} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

Where Q is considered as a simplest form which is defined in section 4.1.

It then follows from theorem 1 that the optimal Laplacian matrix $\mathcal{L}$ is given by (68):

$$\mathcal{L} = \begin{bmatrix} 1.1547 & -0.5774 & -0.5774 \\ -0.5774 & 1.1547 & -0.5774 \\ -0.5774 & -0.5774 & 1.1547 \end{bmatrix}, \tag{68}$$

Subsequently, the solution of algebraic Riccati equation (26) for yaw angle (ψ) of equation (65), with $\mathrm{L} = 2$ (derivation order), is given by (69)

$$\mathrm{P} = \begin{bmatrix} 2.9 & -1.4 & -1.4 & 1.1 & -0.5 & -0.5 \\ -1.4 & 2.9 & -1.4 & -0.5 & 1.1 & -0.5 \\ -1.4 & -1.4 & 2.9 & -0.5 & -0.5 & 1.1 \\ 1.1 & -0.5 & -0.5 & 1.6 & -0.8 & -0.8 \\ -0.5 & 1.1 & -0.5 & -0.8 & 1.6 & -0.8 \\ -0.5 & -0.5 & 1.1 & -0.8 & -0.8 & 1.6 \end{bmatrix}, \tag{69}$$

The following figures show the consensus protocols implemented on a formation 3 quadrotors for their heading states while other independent states viz. 3 position states are regulated for a stable hover via PID controller.

Figure 2 shows initial arbitrary heading values of 64, 110 and 22 degrees for 1st, 2nd and 3rd quadrotors respectively as written near each drone in the flight plan screen.

Figure 3 shows consensus reached value of around 67 degrees for 3 quadrotors with a variation of 1 degree due to magnetometer sensor imperfections, communication delays and other sources of errors. The diagram below the flight plan screen shows the time history of heading trajectories that leads to the consensus value. The planar trajectories of quadrotors in X-Y plane is also depicted in the flight plan screen.

7. Conclusion And Future Works

In this paper, LQR-based optimal consensus algorithms for multiagent systems with L^{th} -order integrator dynamics in contineous-time setting was studied. The basic LQR consensus algorithm was then developed by introducing intenal controller on states derivatives to keep the agreed state values invarient. Stability of the optimal consensus protocol was investigated via Lyapunov function stability criterion. To validate the effectiveness of the expressed consensus protocol, a seccond order consensus algorithm was implemented for heading angles of a set of 3 AR Drone 2 quadrotors using Paparazzi autopilot software.

Future work will be dedicated to include position states in the consensus algorithm experiment.

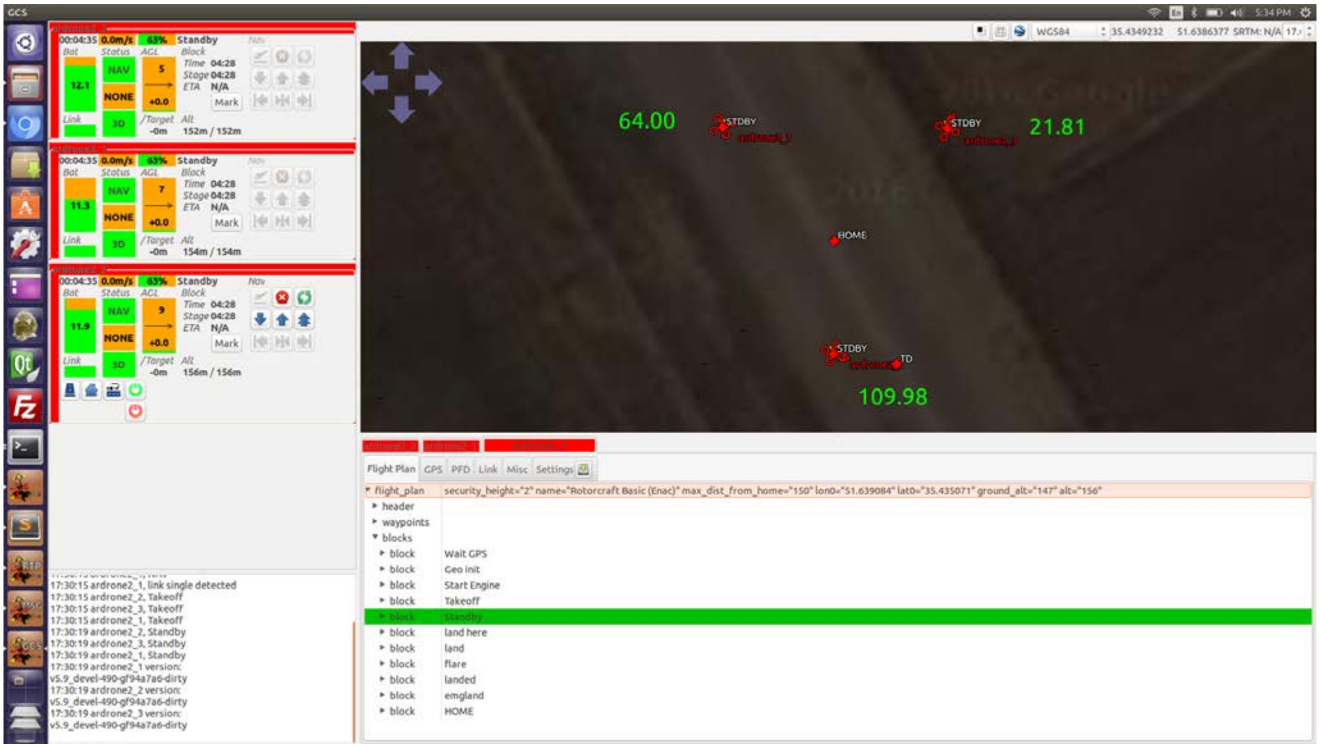

Figure 2. Initial heading values for 3 quadrotors

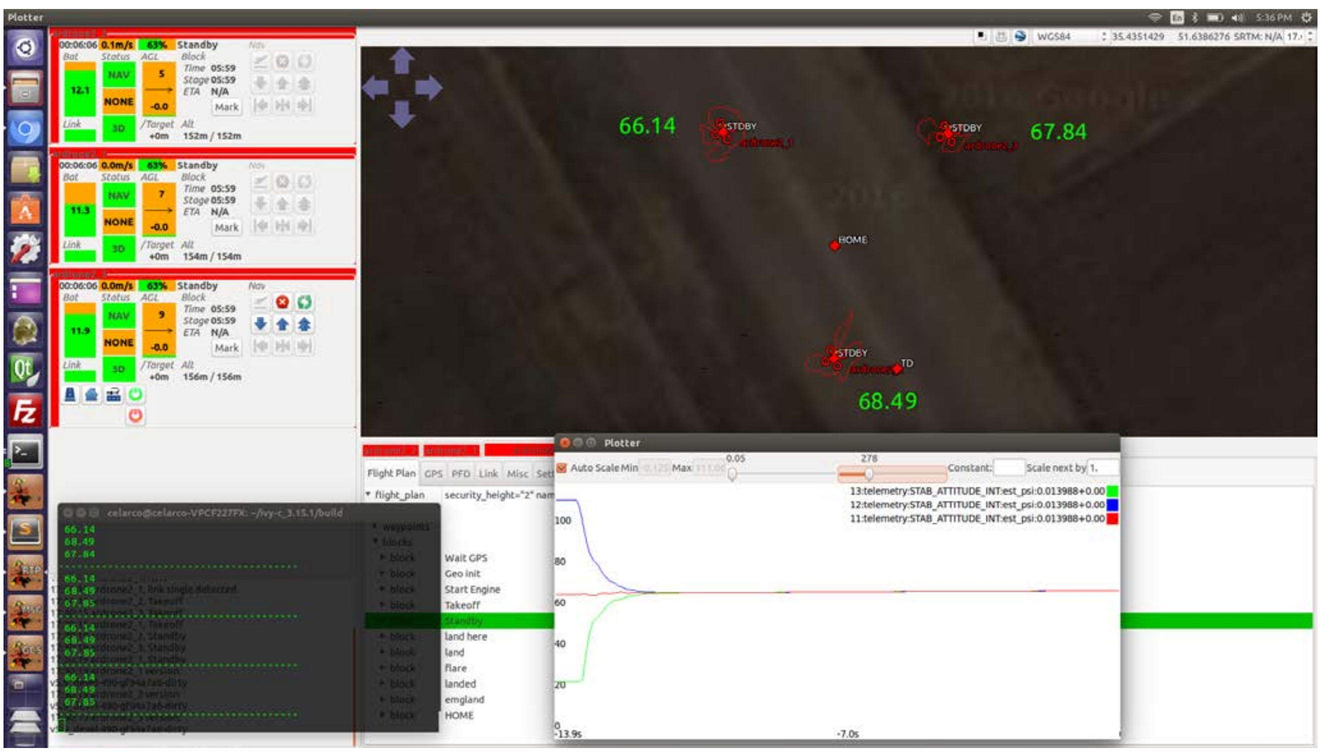

Figure 3. Final consensus reached heading values for 3 quadrotors along with their planar trajectories

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