Mixed differential game in Target-Attacker-Defender problem

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ABSTRACT

This paper addresses a mixed differential game with three players: Target, Attacker and Defender, where the Attacker aims to capture the Target whilst avoiding being captured by the Defender. At this point, there are two pursuit-evasion problems in this game, and two focuses should be considered: one is the cooperation between the Target and the Defender; the other one is the role changes between pursuer and evader of the Attacker. This paper discusses the mixed differential game analytically, obtains the optimal strategies of the three players, and provides numerical solutions for different initial states.

1 INTRODUCTION

Multi-agent pursuit-evasion games has become a research hotpot in the fields of aerospace, control, and robotics. In this paper, we consider a particular differential game, called Target-Attacker-Defender game, where the Attacker chases the Target while avoids being captured by the Defender. Oppositely, the Target should cooperate with the Defender in order to escape from the Attacker whilst acting a bait to help the Defender capture the Attacker. Thus, there are two pursuit-evasion problems in this game: Attacker-Target and Defender-Attacker, so we call it as a mixed differential game.

The literature contains a large number of work on pursuit-evasion games, including the problem formulations and solution techniques of single pursuer-single evader problem [1–3] , and the determination of the role assignments of pursuer and evader in [4, 5].

Pursuit-evasion games with two players cooperating to antagonize to the other player are also covered in the literature [6–8]. For example, a so-called ”fishing game” with point capture is proposed in [6]. The authors construct a barrier by using the method of explicit policy in the game of kind and solve the players' optimal strategies in terms of the game of degree. Cooperation between two agents with the goal of avoiding captured by a single pursuer is addressed in [7] where the agents' strategies should trade off evasion and herding. In [8], inspired by hunting and foraging behaviors of various fish species, the authors present a novel multi-phase cooperative strategy in which the pursuers move in specific formations and confine the evader to a bounded region.

Recently, the Target-Attacker-Defender problem has been concerned by Gutman [9, 10], Shima [11, 12], David and Pachter [13–16]. The problem without cooperation between the Target and the Defender is addressed in [9, 10] where linearization-based Attacker maneuvers are presented to evade the Defender and continue pursuing the Target. Different types of the cooperation between the Target and the Defender are considered in [11–18], where the Target represents an aircraft trying to evade an Attacker and the Defender is a pursuer in order to intercept the Attacker. These articles only consider on target set that the interception point between the Defender and the Attacker is closer to the Target's position. In other words, the Attacker only aims to capture the Target but not regard the threat from the Defender. And the game terminates only when the Target is captured by the Attacker.

In this paper, we consider two focuses on the Target-Attacker-Defender problem: one is the cooperation between the Target and the Defender; the other one is the role changes between pursuer and evader of the Attacker. Specifically, as a pursuer, the Attacker attempts to capture the Target; while as a evader, the Attacker should avoid being captured by the Defender. We obtain the analytical solutions of this mixed differential game, and derive out the optimal strategies of the three players. In addition, we provide the numerical solution of this game for different initial states.

This paper is organized as follows. Section II describes the engagement scenario. In section III optimal control strategies are achieved. In section IV, numerical solutions of the differential game are provided and the examples are given. Finally concluding remarks are made in section V.

2 THE PROBLEM STATEMENT

The Target-Attacker-Defender differential game is shown in Figure 1. The speeds of the Target, Attacker, and Defender are denoted by V_T, V_A, V_D, respectively, which are assumed to be constant. The dynamics of the Target-Attacker-Defender in the realistic game space are given as follows:

$$\dot{x}_T = V_T \cos\hat{\phi}, \dot{y}_T = V_T \sin\hat{\phi} \quad (1)$$

$$\dot{x}_A = V_A \cos\hat{\chi}, \dot{y}_A = V_A \sin\hat{\chi} \quad (2)$$

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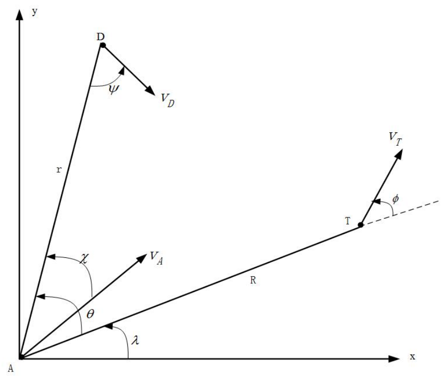

Figure 1: Reduced state space

$$\dot{x}_D = V_D \cos\hat{\psi}, \dot{y}_D = V_D \sin\hat{\psi} \tag{3}$$

where the $\hat{\phi} = \phi + \lambda$, $\hat{\chi} = \lambda + \theta - \chi$, and $\hat{\psi} = \psi + \theta + \lambda - \pi$.

The speed ratio parameters are defined as $\alpha = \frac{V_T}{V_A}$ and $\beta = \frac{V_D}{V_A}$. In this paper, we consider that the speed of the Attacker is faster than the Target and the Defender, so that $\alpha, \beta < 1$. The parameter R_{AT} represents the distance between the Target and the Attacker. The parameter r_{DA} represents the distance between the Attacker and Defender. The positive constant R_c represents the capture radius of the Attacker and the positive constant r_c represents the capture radius of the Defender. This game contains two pursuit-evasion games: the Attacker pursues the Target and the Defender pursues the Attacker. As a result, the Attacker plays two roles, purser for the Target and evader for the Defender. Naturally, in order to win the game as far as possible, the Target and the Defender should cooperate as a team.

3 The Differential Game

In this section, we describe the corresponding dynamics of the three-player engagement using a reduced state space formed by the ranges R_{AT} and r_{DA}, and the angle between them , denoted by θ. The objective of the Attacker is to choose an optimal heading angle, denoted by χ, such that the distance R_{AT} is minimum and the distance r_{DA} is maximum. On the contrary, the objectives of the Target and the Defender are to adopt the optimal heading angles, denoted by ϕ and ψ respectively, such that R_{AT} is maximum and r_{DA} is minimum. When the distance $R_{AT}(t_1) = R_c$ or the distance $r_{DA}(t_2) = r_c$, the differential game terminates. The t_f, denotes the game terminal time, is equal to $\min\{t_1, t_2\}$.

We transform the relative heading angles to the heading angles in the reduced state space with respect to the fixed coordinate axis x using the line of sight angle from the Attacker to the Target, denoted by λ. For the convenience of notation, we denote R_{AT} as R and denote r_{DA} as r. The dynamics in the reduced state space are as follows:

$$\dot{R} = \alpha \cos\phi - \cos(\theta - \chi) \tag{4}$$

$$\dot{r} = -\cos\chi - \beta\cos\psi \tag{5}$$

$$\dot{\theta} = -\frac{\alpha}{R}\sin\phi + \frac{1}{R}sin(\theta-\chi) - \frac{\beta}{r}sin\psi + \frac{1}{r}sin\chi \tag{6}$$

$$\begin{aligned} R(t_0) &= R_0 \\ r(t_0) &= r_0 \\ \theta(t_0) &= \theta_0 \end{aligned} \tag{7}$$

The objective of the Attacker is to minimize the distance R maximize the distance r at the terminal time t_f, where t_f is free. The objective of the Target and the Defender is to maximize the R and minimize the r. Thus, the payoff function can be indicated as

$$\max_{\phi,\psi}\min_{\chi} J = \max_{\phi,\psi}\min_{\chi} \int_{t_0}^{t_f} w_1\dot{R} - w_2\dot{r}dt \tag{8}$$

where the parameters w_1 and w_2 are the weighting coefficients.

Then from the differential game theory [1] [2], the Hamiltonian function is given by

$$\begin{aligned} &H(\lambda,\phi,\psi,\chi,t) = w_1\dot{R} - w_2\dot{r} + \lambda_R\dot{R} + \lambda_\theta\dot{\theta} \\ &= w_1(\alpha\cos\phi - \cos(\theta-\chi)) - w_2(-\cos\chi - \beta\cos\psi) \\ &+\lambda_R(\alpha\cos\phi - \cos(\theta-\chi)) + \lambda_r(-\cos\chi - \beta\cos\psi) \\ &+\lambda_\theta(-\frac{\alpha}{R}\sin\phi + \frac{1}{R}\sin(\theta-\chi) - \frac{\beta}{r}\sin\psi + \frac{1}{r}\sin\chi) \end{aligned} \tag{9}$$

The co-state dynamics are given by

$$\dot{\lambda}_R = \frac{\lambda_\theta}{R^2}(sin(\theta-\chi) - \alpha sin\phi) \tag{10}$$

$$\dot{\lambda}_r = \frac{\lambda_\theta}{r^2}(sin\chi - \beta sin\psi) \tag{11}$$

$$\dot{\lambda}_\theta = -w_1\sin(\theta-\chi) - \lambda_R\sin(\theta-\chi) - \frac{\lambda_\theta}{R}\cos(\theta-\chi) \tag{12}$$

The game ends in two cases. In one case, we define the team of the Defender and the Target wins the game and the Attacker is captured. The terminal state $r(t_f)$ is fixed and equal to r_c. The terminal states $R(t_f)$ and $\theta(t_f)$ are free, we define $\lambda_R(t_f) = 0, \lambda_\theta(t_f) = 0$. The best solution for this problem requires that the terminal condition $H(x^*(t_f), u^*(t_f), \lambda^*(t_f), t_f) = 0$. In summary, the terminal conditions are:

$$\begin{aligned} &r(t_f) = r_c \\ &\lambda_R(t_f) = 0 \\ &\lambda_\theta(t_f) = 0 \\ &w_1\alpha + \beta(w_2 - \lambda_r(t_f)) \\ &-\sqrt{w_1^2 + (w_2 - \lambda_r(t_f))^2 - 2w_1(w_2 - \lambda_r(t_f))cos\theta(t_f)} = 0 \end{aligned} \tag{13}$$

In the other case, the Attacker wins the game, captures the Target but not be captured by the Defender, the terminal conditions are:

$$\begin{gathered} R(t_f) = R_c \\ \lambda_r(t_f) = 0 \\ \lambda_\theta(t_f) = 0 \\ (w_1 + \lambda_R(t_f))\alpha + w_2\beta \\ -\sqrt{(w_1 + \lambda_R(t_f))^2 + {w_2}^2 - 2w_2(w_1 + \lambda_R(t_f))cos\theta(t_f)} = 0 \end{gathered} \tag{14}$$

Theorem 1: In the mixed differential game, the Target and Defender optimal control angles are given by

$$\begin{aligned} \sin\phi^* &= -\frac{\lambda_\theta/R}{\sqrt{(\lambda_\theta/R)^2 + (w_1+\lambda_R)^2}} \\ \cos\phi^* &= \frac{w_1+\lambda_R}{\sqrt{(\lambda_\theta/R)^2 + (w_1+\lambda_R)^2}} \end{aligned} \tag{15}$$

$$\begin{aligned} \sin\psi^* &= -\frac{\lambda_\theta/r}{\sqrt{(\lambda_\theta/r)^2 + (w_2-\lambda_r)^2}} \\ \cos\psi^* &= \frac{w_2-\lambda_r}{\sqrt{(\lambda_\theta/r)^2 + (w_2-\lambda_r)^2}} \end{aligned} \tag{16}$$

The optimal control angle of the Attacker is given by E-qs(17)(shown on the top of the next page).

Proof: The analytic solution of differential game can be obtained by:

(1) Differentiating the Hamiltonian function in ϕ and setting the derivative to zero:

$$\frac{\partial H}{\partial \phi} = \alpha[-\frac{\lambda_\theta}{R}cos\phi - (w_1+\lambda_R)sin\phi] = 0 \tag{18}$$

Using the trigonometric identity

$$\sin^2\phi + \cos^2\phi = 1$$

we can conclude Eqs(15).

The second partial derivative of the Hamiltonian function in ϕ :

$$\begin{aligned} \frac{\partial^2 H}{\partial \phi^2} &= \alpha[\frac{\lambda_\theta}{R}\sin\phi - (w_1+\lambda_R)cos\phi] = \\ &-\alpha[\frac{\lambda_\theta^2 + R^2(w_1+\lambda_R)^2}{R^2\sqrt{(\lambda_\theta/R)^2 + (w_1+\lambda_R)^2}}] < 0 \end{aligned} \tag{19}$$

which means that ϕ^* maximizes the payoff J; That is to say, it maximizes the final distance $R(t_f)$ and minimizes the distance $r(t_f)$.

(2) With regard to ψ ,we can obtain the optimal heading of the Defender in a similar way.

$$\frac{\partial H}{\partial \psi} = \beta[-\frac{\lambda_\theta}{r}cos\psi - (w_2-\lambda_r)sin\psi] = 0 \tag{20}$$

and conclude Eqs(16).

The second partial derivative of the Hamiltonian function in ψ :

$$\begin{aligned} \frac{\partial^2 H}{\partial \psi^2} &= \beta[\frac{\lambda_\theta}{r}\sin\psi - (w_2-\lambda_r)cos\psi] = \\ &-\beta[\frac{\lambda_\theta^2 + r^2(w_2-\lambda_r)^2}{r^2\sqrt{(\lambda_\theta/r)^2 + (w_2-\lambda_r)^2}}] < 0 \end{aligned} \tag{21}$$

which means that ψ^* maximizes the payoff J; that is to say, it maximizes the final distance $R(t_f)$ and minimizes the distance $r(t_f)$.

(3) The optimal heading χ^* of the Attacker can also be solved by differentiating the Hamiltonian function in χ and setting the derivative to zero:

$$\begin{aligned} \frac{\partial H}{\partial \chi} &= \left((w_1+\lambda_R)cos\theta - (w_2-\lambda_r) - \frac{\lambda_\theta}{R}\sin\theta\right)\sin\chi \\ &-\left((w_1+\lambda_R)sin\theta + \frac{\lambda_\theta}{R}\cos\theta - \frac{\lambda_\theta}{r}\right)\cos\chi = 0 \end{aligned} \tag{22}$$

Letting

$$a = (w_1+\lambda_R)sin\theta + \frac{\lambda_\theta}{R}\cos\theta - \frac{\lambda_\theta}{r}$$

$$b = (w_1+\lambda_R)cos\theta - (w_2-\lambda_r) - \frac{\lambda_\theta}{R}\sin\theta$$

Then, we can obtain

$$\begin{aligned} \sin\chi^* &= \frac{a}{\sqrt{a^2+b^2}} \\ \cos\chi^* &= \frac{b}{\sqrt{a^2+b^2}} \end{aligned} \tag{23}$$

Furthermore, we have

$$\frac{\partial^2 H}{\partial \chi^2} = b\cos\chi + a\sin\chi = \frac{b^2}{\sqrt{a^2+b^2}} + \frac{a^2}{\sqrt{a^2+b^2}} > 0 \tag{24}$$

It shows that the optimal heading of the Attacker minimizes the payoff J, which is equivalent to minimizing the distance $R(t_f)$ and maximizing the distance $r(t_f)$.

$$\sin \chi^* = \frac{(w_1 + \lambda_R)sin\theta + \frac{\lambda_\theta}{R}\cos\theta - \frac{\lambda_\theta}{r}}{\sqrt{\left((w_1 + \lambda_R)sin\theta + \frac{\lambda_\theta}{R}\cos\theta - \frac{\lambda_\theta}{r}\right)^2 + \left((w_1 + \lambda_R)cos\theta - (w_2 - \lambda_r) - \frac{\lambda_\theta}{R}\sin\theta\right)^2}}$$

$$\cos \chi^* = \frac{(w_1 + \lambda_R)cos\theta - (w_2 - \lambda_r) - \frac{\lambda_\theta}{R}\sin\theta}{\sqrt{\left((w_1 + \lambda_R)sin\theta + \frac{\lambda_\theta}{R}\cos\theta - \frac{\lambda_\theta}{r}\right)^2 + \left((w_1 + \lambda_R)cos\theta - (w_2 - \lambda_r) - \frac{\lambda_\theta}{R}\sin\theta\right)^2}} \tag{17}$$

4 Numerical Solutions of The Optimal Control Problem

Eqs(4)-(14) represent a two point boundary value problem (TPBVP). In many cases, it is difficult to solve analytically. At present, there are some methods to solve this problem numerically [19], such as shooting method, difference method and finite element method and so on. Regardless of the methods, it is necessary to estimate the initial values of the TPBVP. If the initial values are uncertain, obtaining an ideal solution will be very difficult. That is the main challenge in solving TPBVP [20]. Since the co-state variable has no obvious physical meaning, in this paper, we indirectly estimate the initial value of the co-state variables by setting the control variables. Considering the cooperation of the Target and the Defender, we assume that at the initial time the Target moves toward the Defender and the Defender moves toward the Target. The Attacker, aimed to capture the Target, moves toward the Target initially.

Example1: Let the initial conditions be given by $R_0 = 5, r_0 = 5$, and $\theta_0 = 1rad$ .The initial line-of-sight angle is given by $\lambda_0 = 1rad$. The Attacker initially located at the coordinates $(0, 0)$. The capture radiuses are given by $R_c = 1$, and $r_c = 1$. The speed ratios are $\alpha = 0.6$, and$\beta = 0.8$. We obtain the initial value of the co-state variables using the initial value of the control angles: $\lambda_R(0) = 0.2235, \lambda_r(0) = -0.2235$, and $\lambda_\theta(0) = -6.6215$ .The optimal control strategies of three players are shown in Figure 2 and the optimal states trajectories of three players are shown in the Figure 3. The optimal motion trajectories of three players are shown in Figure 4. The 'o' and '*' respectively represent the initial point and the terminal point of three players. We noted that the Defender captured the Attacker at 4.6s. That is to say, the Defender and the Target win the game.

Example2: We give the initial conditions $r_0 = 3, \theta_0 = 2.9rad, \lambda_0 = 0.5rad$, and the other parameters are same as the parameters in example 1. Then we have $\lambda_R(0) = 34.7858, \lambda_r(0) = -37.1870$, and $\lambda_\theta(0) = -17.2051$. The optimal control strategies and the optimal states trajectories are shown in Figure 5 and Figure 6, respectively. The optimal motion trajectories of three players are shown in Figure 7. We noted that the Attacker captured the Target at 10.5s. That is to say, the Attacker win the game.

Example3: Based on the initial conditions of example1,we change the initial conditions $r_0 = 3, \theta_0 = 1.5rad$, and $\lambda_0 = 0.5rad$. The results of game are shown in Figure 8, Figure 9 and Figure 10. We noted that the game continues at all times and no one can win the game.

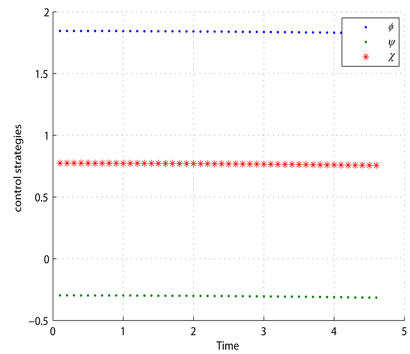

Figure 2: The optimal heading angles of three players in Example1

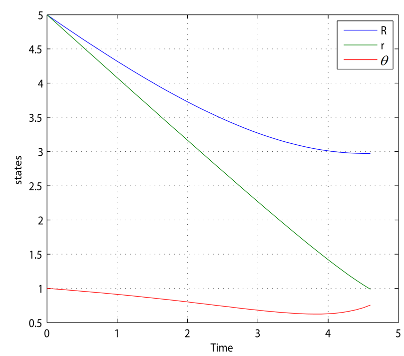

Figure 3: The optimal state trajectories of three players in Example1

5 Conclusions

In this paper, we formulate an Attacker-Target-Defender mixed differential game and solve the optimal solutions. In this game, two pursuit-evasion games are described and two target sets are determined. The Attacker plays two roles, and the Target and the Defender cooperate with each other. We provide the optimal strategy for each player. By estimating

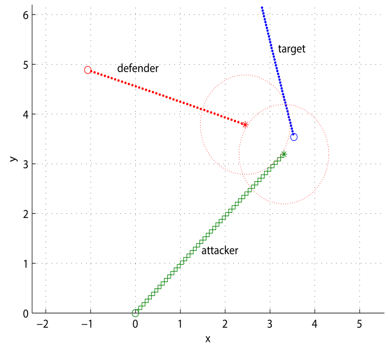

Figure 4: The optimal motion trajectories of three players in Example1

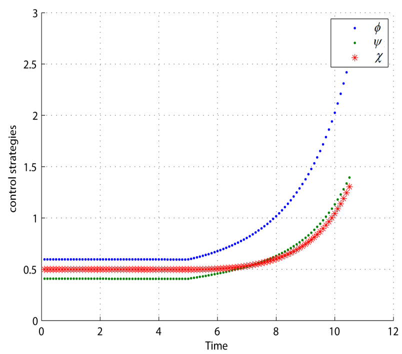

Figure 5: The optimal heading angles of three players in Example2

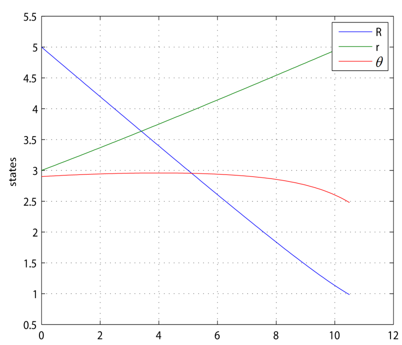

Figure 6: The optimal state trajectories of three players in Example2

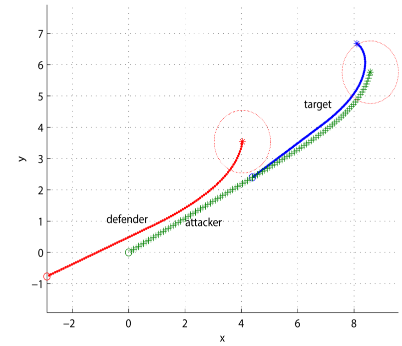

Figure 7: The optimal motion trajectories of three players in Example2

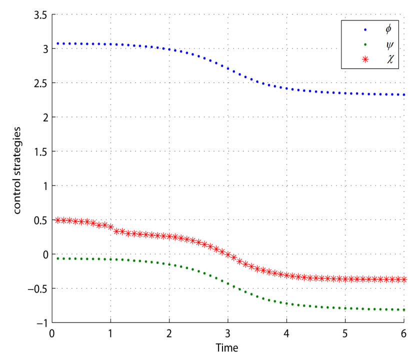

Figure 8: The optimal heading angles of three players in Example3

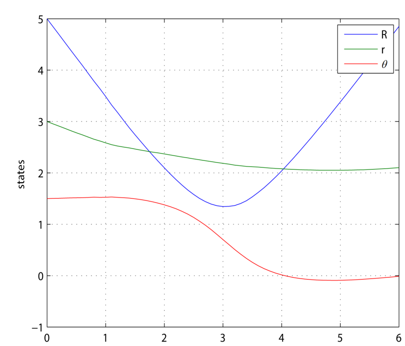

Figure 9: The optimal state trajectories of three players in Example3

the initial value of the control variables, we obtain the numerical solutions of this problem and illustrate the simulation results. In different initial conditions, three players adopt this optimal strategies may cause different outcome: the Attacker or the team of Target and Defender wins the game, even more no one wins the game. In the further, we will analyze the winning conditions for each player involving the initial position,speed ratio, capture radius and the weighting coefficients. Also, we will construct such a barrier to delineate the winning region for each player and obtain a complete solution for this game.

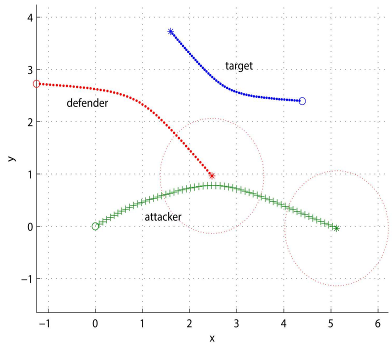

Figure 10: The optimal motion trajectories of three players in Example3

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